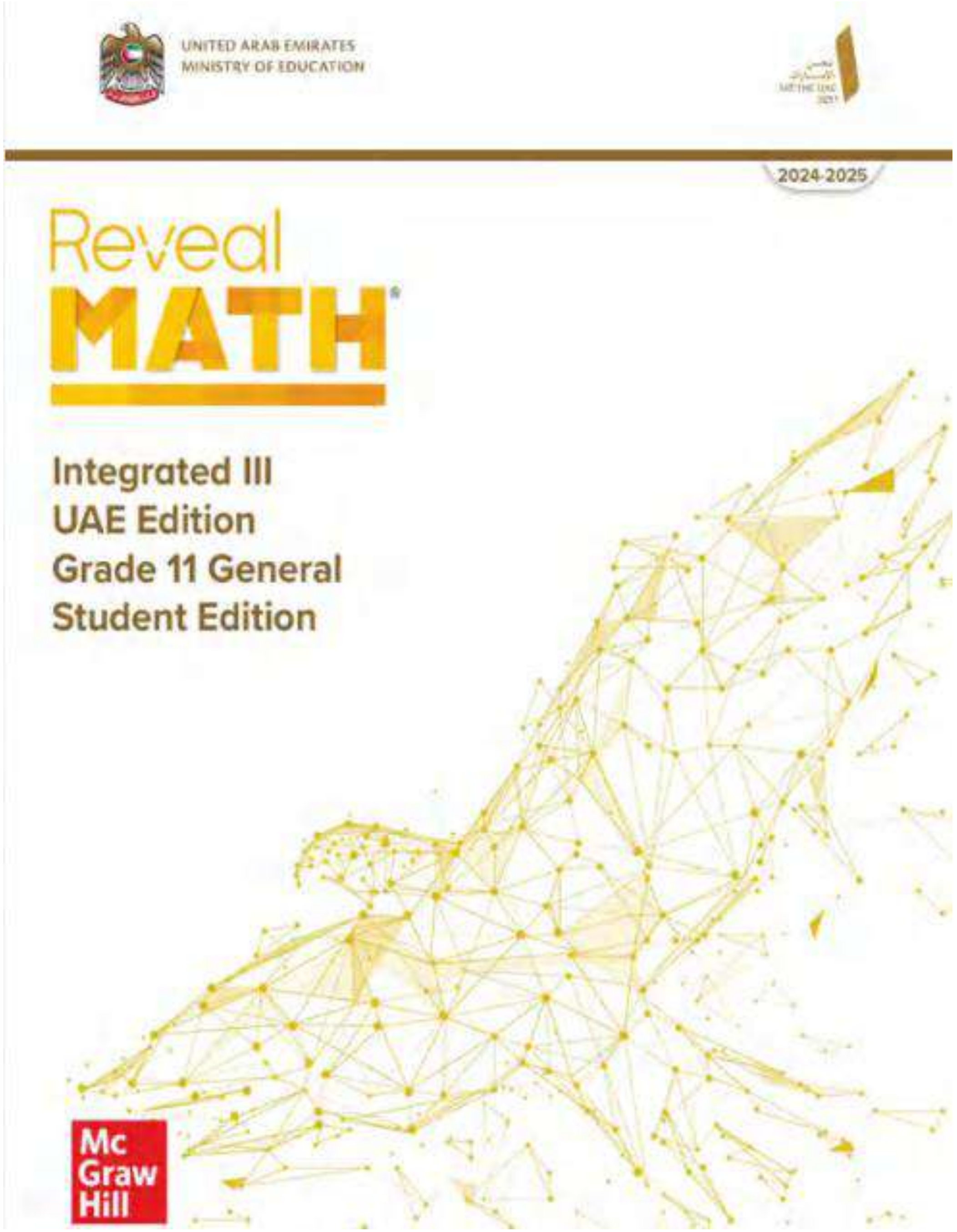
حل تجميعة أسئلة وفق الهيكل الوزاري منهج ريفيل مع تدريبات

إعداد: AlSabhi Abdulaziz Saif



11-general-Math-EOT ✿

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Subject المادة	Mathematics/Reveal الرياضيات/ريفيل
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Number of MCQ عدد الأسئلة الموضوعية	15
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FRQ - الأسئلة المقالية	1	Simplify expressions in exponential or radical form	1-12 & 27-47	179-180
		Write expressions with rational exponents in radical form and vice versa	13-18	179
	2	Solve radical equations	13-20 & 31-42	207-208
	3	Find sums of geometric series	13-29	245-246
	4	Simplify rational expressions by multiplying and dividing	24-35	316
	5	Graph rational functions with vertical and horizontal asymptotes	1-10; Example 1 & 3	343; 337-338-339-340
MCQ - الأسئلة الموضوعية	6	Find the composition of functions	21-35	165
	7	Find the inverse of a function or relation	5-14	171
	8	Write expressions with rational exponents in radical form and vice versa	1-12	179
	9	Graph and analyze radical functions	27-39	191
	10	Add, subtract, multiply, and divide radical expressions	29-38	200
	11	Solve radical equations	13-20	207
	12	Graph growth and decay functions	1-13	221
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14	Evaluate expressions involving the natural base and natural logarithm	10-15	237
15	Simplify rational expressions	1-16	315
16	Simplify complex algebraic fractions including rational expressions	16-19	323
17	Determine properties of reciprocal functions	51-56	336
18	Graph rational functions with oblique asymptotes and point discontinuity	11-30	344
19	Recognize and solve direct and joint variation problems	19-22	351
20	Solve rational equations	1-12	361

الاسئلة الكتابيه للامتحان

FRQ - الأسئلة المقالية				
	1	Simplify expressions in exponential or radical form	1-12 & 27-47	179-180
		Write expressions with rational exponents in radical form and vice versa	13-18	179
	2	Solve radical equations	13-20 & 31-42	207-208
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1	Simplify expressions in exponential or radical form	1-12 & 27-47	179-180
	Write expressions with rational exponents in radical form and vice versa	13-18	179

Examples 1 and 2

Simplify.

1. $\pm\sqrt{121x^4y^{16}}$

2. $\pm\sqrt{225a^{16}b^{36}}$

3. $\pm\sqrt{49x^4}$

4. $-\sqrt{16c^4d^2}$

5. $-\sqrt{81a^{16}b^{20}c^{12}}$

6. $-\sqrt{400x^{32}y^{40}}$

7. $\sqrt[4]{16(x-3)^{12}}$

8. $\sqrt[8]{x^{16}y^8}$

1	Simplify expressions in exponential or radical form	1-12 & 27-47	179-180
	Write expressions with rational exponents in radical form and vice versa	13-18	179

9. $\sqrt[4]{81(x-4)^4}$

10. $\sqrt[6]{x^{18}}$

11. $\sqrt[4]{a^{12}}$

12. $\sqrt[3]{a^{12}}$

Examples 3

Write each expression in radical form, or write each radical in exponential form.

13. $8^{\frac{1}{5}}$

14. $4^{\frac{2}{7}}$

15. $(x^3)^{\frac{3}{2}}$

1	Simplify expressions in exponential or radical form	1-12 & 27-47	179-180
	Write expressions with rational exponents in radical form and vice versa	13-18	179

16. $\sqrt{17}$

17. $\sqrt[3]{5xy^2}$

18. $\sqrt[4]{625x^2}$

Example 6

Simplify each expression.

27. $x^{\frac{1}{3}} \cdot x^{\frac{2}{5}}$

28. $a^{\frac{4}{9}} \cdot a^{\frac{1}{4}}$

29. $b^{-\frac{3}{4}}$

30. $y^{-\frac{4}{5}}$

1	Simplify expressions in exponential or radical form	1-12 & 27-47	179-180
	Write expressions with rational exponents in radical form and vice versa	13-18	179

solution method

Lesson 4-3

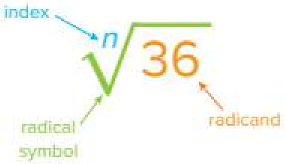
nth Roots and Rational Exponents

Learn *n*th Roots

Finding the square root of a number and squaring a number are inverse operations. To find the square root of a, you must find a number with a square of a. The inverse of raising a number to the nth power is finding the ***n*th root** of a number. The symbol $\sqrt[n]{\ }$ indicates an nth root.

For any real numbers a and b and any positive integer n, if $a^n = b$, then a is an nth root of b. For example, because $(-2)^6 = 64$, -2 is a sixth root of 64 and 2 is a principal root.

An example of an nth root is $\sqrt[n]{36}$, which is read as *the nth root of 36*. In this example, n is the **index** and 36 is the **radicand**, or the expression under the radical symbol.



Some numbers have more than one real nth root. For example, 16 has two square roots, 4 and -4, because 4^2 and $(-4)^2$ both equal 16. When there is more than one real root and n is even, the nonnegative root is called the **principal root**.

Key Concept • Real *n*th Roots

Suppose n is an integer greater than 1, a is a real number, and a is an nth root of b.

a	n is even.	n is odd.
$a > 0$	1 unique positive and 1 unique negative real root: $\pm\sqrt[n]{a}$	1 unique positive and 0 negative real root: $\sqrt[n]{a}$
$a < 0$	0 real roots	0 positive and 1 negative real root: $\sqrt[n]{a}$
$a = 0$	1 real root: $\sqrt[n]{0} = 0$	1 real root: $\sqrt[n]{0} = 0$

A radical expression is simplified when the radicand contains no fractions and no radicals appear in the denominator.

Example 1 Find Roots

Example 2 Simplify Using Absolute Value

Learn Rational Exponents

You can use the properties of exponents to translate expressions from exponential form to radical form or from radical form to exponential form. An expression is in **exponential form** if it is in the form x^n, where n is an exponent. An expression is in **radical form** if it contains a radical symbol.

For any real number b and a positive integer n, $b^{\frac{1}{n}} = \sqrt[n]{b}$, except where $b < 0$ and n is even. When $b < 0$ and n is even, a complex root may exist.

Examples: $125^{\frac{1}{3}} = \sqrt[3]{125}$ or 5 $\qquad (-49)^{\frac{1}{2}} = \sqrt{-49}$ or $7i$

The expression $b^{\frac{1}{n}}$ has a **rational exponent**. The rules for exponents also apply to rational exponents.

Key Concept • Rational Exponents

For any nonzero number b and any integers x and y, with $y > 1$, $b^{\frac{x}{y}} = \sqrt[y]{b^x} = \left(\sqrt[y]{b}\right)^x$, except when $b < 0$ and y is even. When $b < 0$ and y is even, a complex root may exist.

Examples: $125^{\frac{2}{3}} = \left(\sqrt[3]{125}\right)^2 = 5^2$ or 25

$(-49)^{\frac{3}{2}} = (\sqrt{-49})^3 = (7i)^3$ or $-343i$

Key Concept • Simplest Form of Expressions with Rational Exponents

An expression with rational exponents is in simplest form when all of the following conditions are met.

- It has no negative exponents.
- It has no exponents that are not positive integers in the denominator.
- It is not a complex fraction.
- The index of any remaining radical is the least number possible.

Example 3 Radical and Exponential Forms

Example 6 Simplify Expressions with Rational Exponents

Solution

Simplify.

1. $\pm\sqrt{121x^4y^{16}}$

SOLUTION:

$$\pm\sqrt{121x^4y^{16}} = \pm\sqrt{(11x^2y^8)^2}$$
$$= \pm 11x^2y^8$$

ANSWER:

$\pm 11x^2y^8$

2. $\pm\sqrt{125a^{16}b^{36}}$

SOLUTION:

$$\pm\sqrt{125a^{16}b^{36}} = \pm\sqrt{(15a^8b^{18})^2}$$
$$= \pm 15a^8b^{18}$$

ANSWER:

$\pm 15a^8b^{18}$

3. $\pm\sqrt{49x^4}$

SOLUTION:

$$\pm\sqrt{49x^4} = \pm\sqrt{(7x^2)^2}$$
$$= \pm 7x^2$$

ANSWER:

$\pm 7x^2$

1	Simplify expressions in exponential or radical form	1-12 & 27-47	179-180
	Write expressions with rational exponents in radical form and vice versa	13-18	179

Solution

4. $-\sqrt{16c^4d^2}$

SOLUTION:

$$-\sqrt{16c^4d^2} = -\sqrt{(4c^2d)^2} = -4c^2|d|$$

Since d could be negative, you must use the absolute value of d to ensure that the principal square root is nonnegative.

ANSWER:

$-4c^2|d|$

5. $-\sqrt{81a^{16}b^{20}c^{12}}$

SOLUTION:

$$-\sqrt{81a^{16}b^{20}c^{12}} = -\sqrt{(9a^8b^{10}c^6)^2} = -9a^8b^{10}c^6$$

ANSWER:

$-9a^8b^{10}c^6$

6. $-\sqrt{400x^{32}y^{40}}$

SOLUTION:

$$-\sqrt{400x^{32}y^{40}} = -\sqrt{(20x^{16}y^{20})^2} = -20x^{16}y^{20}$$

ANSWER:

$-20x^{16}y^{20}$

7. $\sqrt[4]{16(x-3)^{12}}$

SOLUTION:

$$\sqrt[4]{16(x-3)^{12}} = \sqrt[4]{16}\cdot\sqrt[4]{(x-3)^{12}} = 2|(x-3)^3|$$

Since $(x-3)^3$ could be negative, you must use the absolute value of $(x-3)^3$ to ensure that the principal square root is nonnegative.

ANSWER:

$2|(x-3)^3|$

8. $\sqrt[8]{x^{16}y^8}$

SOLUTION:

$$\sqrt[8]{x^{16}y^8} = \sqrt[8]{(x^2y)^8} = x^2|y|$$

Because y could be negative, you must use the absolute value of y to ensure that the principal root is nonnegative.

ANSWER:

$x^2|y|$

9. $\sqrt[4]{81(x-4)^4}$

SOLUTION:

$$\sqrt[4]{81(x-4)^4} = \sqrt[4]{81}\cdot\sqrt[4]{(x-4)^4} = 3|x-4|$$

Since $(x-4)$ could be negative, you must use the absolute value of $(x-4)$ to ensure that the principal square root is nonnegative.

ANSWER:

$3|x-4|$

10. $\sqrt[6]{x^{18}}$

SOLUTION:

$$\sqrt[6]{x^{18}} = \sqrt[6]{(x^3)^6} = |x^3|$$

Because x^3 could be negative, you must use the absolute value of x^3 to ensure that the principal root is nonnegative.

ANSWER:

$|x^3|$

11. $\sqrt[4]{a^{12}}$

SOLUTION:

$$\sqrt[4]{a^{12}} = \sqrt[4]{(a^3)^4} = |a^3|$$

Because a^3 could be negative, you must use the absolute value of a^3 to ensure that the principal root is nonnegative.

ANSWER:

$|a^3|$

12. $\sqrt[3]{a^{12}}$

SOLUTION:

$$\sqrt[3]{a^{12}} = \sqrt[3]{(a^4)^3} = a^4$$

ANSWER:

a^4

Write each expression in radical form, or write each radical in exponential form.

13. $8^{\frac{1}{5}}$

SOLUTION:

$8^{\frac{1}{5}} = \sqrt[5]{8^1}$

$= \sqrt[5]{8}$

ANSWER:

$\sqrt[5]{8}$

14. $4^{\frac{2}{7}}$

SOLUTION:

$4^{\frac{2}{7}} = \sqrt[7]{4^2}$

$= \sqrt[7]{16}$

ANSWER:

$\sqrt[7]{16}$

15. $(x^3)^{\frac{3}{2}}$

SOLUTION:

$(x^3)^{\frac{3}{2}} = x^{\frac{9}{2}}$

$= \sqrt{x^9}$

ANSWER:

$\sqrt{x^9}$

16. $\sqrt{17}$

SOLUTION:

$\sqrt{17} = 17^{\frac{1}{2}}$

ANSWER:

$17^{\frac{1}{2}}$

17. $\sqrt[3]{5xy^2}$

SOLUTION:

$\sqrt[3]{5xy^2} = 5^{\frac{1}{3}} x^{\frac{1}{3}} y^{\frac{2}{3}}$

ANSWER:

$5^{\frac{1}{3}} x^{\frac{1}{3}} y^{\frac{2}{3}}$

18. $\sqrt[4]{625x^2}$

SOLUTION:

$\sqrt[4]{625x^2} = 625^{\frac{1}{4}} x^{\frac{2}{4}}$

$= 5x^{\frac{1}{2}}$

ANSWER:

$5x^{\frac{1}{2}}$

Simplify each expression.

27. $x^{\frac{1}{3}} \bullet x^{\frac{2}{5}}$

SOLUTION:

$x^{\frac{1}{3}} \bullet x^{\frac{2}{5}} = x^{\frac{1}{3}+\frac{2}{5}}$ Add powers.

$= x^{\frac{5}{15}+\frac{6}{15}}$ $\frac{1}{3} = \frac{5}{15}, \frac{2}{5} = \frac{6}{15}$

$= x^{\frac{11}{15}}$ Add the exponents.

ANSWER:

$x^{\frac{11}{15}}$

28. $a^{\frac{4}{9}} \bullet a^{\frac{1}{4}}$

SOLUTION:

$a^{\frac{4}{9}} \bullet a^{\frac{1}{4}} = a^{\frac{4}{9}+\frac{1}{4}}$ Add powers.

$= a^{\frac{16}{36}+\frac{9}{36}}$ $\frac{4}{9} = \frac{16}{36}, \frac{1}{4} = \frac{9}{36}$

$= a^{\frac{25}{36}}$ Add the exponents.

ANSWER:

$a^{\frac{25}{36}}$

30. $y^{-\frac{4}{5}}$

SOLUTION:

$$y^{-\frac{4}{5}} = \frac{1}{y^{\frac{4}{5}}} \qquad b^{-n} = \frac{1}{b^n}$$

$$= \frac{1}{y^{\frac{4}{5}}} \cdot \frac{y^{\frac{1}{5}}}{y^{\frac{1}{5}}} \qquad \frac{y^{\frac{1}{5}}}{y^{\frac{1}{5}}} = 1$$

$$= \frac{y^{\frac{1}{5}}}{y^{\frac{5}{5}}} \text{ or } \frac{y^{\frac{1}{5}}}{y} \qquad y^{\frac{4}{5}} \cdot y^{\frac{1}{5}} = y^{\frac{4}{5}+\frac{1}{5}}$$

ANSWER:

$$\frac{y^{\frac{1}{5}}}{y}$$

2	Solve radical equations	13-20 & 31-42	207-208

Examples 3 and 4

Solve each equation. Identify any extraneous solutions.

13. $\sqrt{x-15} = 3 - \sqrt{x}$

14. $(5q + 1)^{\frac{1}{4}} + 7 = 5$

15. $(3x + 7)^{\frac{1}{4}} - 3 = 1$

16. $(3y - 2)^{\frac{1}{5}} + 5 = 6$

2	Solve radical equations	13-20 & 31-42	207-208

Examples 3 and 4

Solve each equation. Identify any extraneous solutions.

17. $(4z - 1)^{\frac{1}{5}} - 1 = 2$

18. $\sqrt{x - 10} = 1 - \sqrt{x}$

19. $\sqrt[6]{y + 2} + 9 = 14$

20. $(2x - 1)^{\frac{1}{4}} - 2 = 1$

2	Solve radical equations	13-20 & 31-42	207-208

Examples 3 and 4

Solve each equation. Identify any extraneous solutions.

31. $6 + \sqrt{4x + 8} = 9$

32. $\sqrt{7a - 2} = \sqrt{a + 3}$

33. $\sqrt{x - 5} - \sqrt{x} = -2$

34. $\sqrt{b - 6} + \sqrt{b} = 3$

2	Solve radical equations	13-20 & 31-42	207-208

Examples 3 and 4

Solve each equation. Identify any extraneous solutions.

35. $2(x-10)^{\frac{1}{3}}+4=0$

36. $3(x+5)^{\frac{1}{3}}-6=0$

37. $\frac{1}{7}(14a)^{\frac{1}{3}}=1$

38. $\frac{1}{4}(32b)^{\frac{1}{3}}=1$

2	Solve radical equations	13-20 & 31-42	207-208

Examples 3 and 4

Solve each equation. Identify any extraneous solutions.

39. $\sqrt{x-3} = 3 - x$

40. $\sqrt{x-2} = 22 - x$

41. $\sqrt{x+30} = x$

42. $\sqrt{x+22} = x + 2$

2	Solve radical equations	13-20 & 31-42	207-208

solution method

Lesson 4-6

Solving Radical Equations

Learn Solving Radical Equations Algebraically

A **radical equation** has a variable in a radicand. When solving a radical equation, the result may be an extraneous solution.

Key Concept • Solving Radical Equations

Step 1 Isolate the radical on one side of the equation.

Step 2 To eliminate the radical, raise each side of the equation to a power equal to the index of the radical.

Step 3 Solve the resulting polynomial equation. Check your results.

Example 3 Identify Extraneous Solutions

Example 4 Solve a Radical Equation

3	Find sums of geometric series	13-29	245-246

Example 3

13. FAMILY Amanda is researching her ancestry. She records names and birth dates for her parents, their parents, and so on, in an online research tool. If she can locate all of the information, how many names will Amanda record in the generation that is 5 generations before her?

14. MOORE'S LAW Gordon Moore, co-founder of Intel, suggested that the number of transistors on a square inch of integrated circuit in a computer chip would double every 18 months. Assuming Moore's law is true, how many times as many transistors would you expect on a square inch of integrated circuit in year 6?

3	Find sums of geometric series	13-29	245-246

Example 4

Write an equation for the *n*th term of each geometric sequence.

15. 3, 9, 27, ...

16. −1, −3, −9, ...

17. 2, −6, 18, ...

18. 5, 10, 20, ...

19. $a_4 = 324$ and $r = 3$

20. $a_3 = 512$ and $r = \frac{1}{8}$

3	Find sums of geometric series	13-29	245-246

Example 5

Given a formula for a geometric sequence in recursive or explicit form, translate it to the other form.

21. $a_1 = 3, a_n = 0.6a_{n-1}, n \geq 2$

22. $a_n = 0.8(2)^{n-1}$

23. $a_1 = -1, a_n = \frac{1}{2}a_{n-1}, n \geq 2$

24. $a_n = -\frac{2}{3}(6)^{n-1}$

3	Find sums of geometric series	13-29	245-246

Example 6

Find the geometric means of each sequence.

25. 4, ___?___, ___?___, ___?___, 64

26. 1, ___?___, ___?___, ___?___, 81

27. 38; 228; ___?___; 8208; 49,248; ...

28. 51; ___?___; 4131; ___?___; 334,611; ...

3	Find sums of geometric series	13-29	245-246

Example 7

29. SCIENTIFIC RESEARCH Scientific balloons carry equipment to observe or conduct experiments. The NASA Balloon Program generally tries to fly balloons above 80,000 to 90,000 feet. Suppose a balloon rises 1000 feet in the first minute after it is launched. For the next hour, each minute it rises 1% more than it rose in the previous minute.

a. Copy and complete the table to show the height of the balloon at various times after launch.

Time (s)	1	2	3	4	5	6
Height (ft)						

b. After an hour will the balloon have reached its target height of 80,000 – 90,000 feet? Explain.

3	Find sums of geometric series	13-29	245-246

solution method

Lesson 5-4

Geometric Sequences and Series

Learn Sequences

A **sequence** is a set of numbers in a particular order or pattern. Each number in a sequence is called a **term**. The first term of a sequence is denoted a_1, the second term is a_2, and so on. A **finite sequence** contains a limited number of terms, while an **infinite sequence** continues without end.

A sequence can be defined as a function.

Key Concept • Sequences as Functions		
Words	A sequence is a function in which the domain consists of natural numbers, and the range consists of real numbers.	
Symbols	Domain: 1 2 3 ... n Range: a_1 a_2 a_3 ... a_n	the position of a term the terms of the sequence

Apply Example 3 Find the *n*th Term

Example 4 Write an Equation for the *n*th Term

Example 5 Recursive and Explicit Formulas

Example 6 Find Geometric Means

Learn Geometric Series

A **series** is the sum of the terms in a sequence. The sum of the first n terms of a series is denoted S_n. A **geometric series** is the sum of the terms of a geometric sequence.

Key Concept • Partial Sums of a Geometric Series	
Given	**The sum of S_n of the first n terms is:**
a_1, r, and n	$S_n = \frac{a_1 - a_1r^n}{1-r}, r \neq 1$
a_1, r, and a_n	$S_n = \frac{a_1 - a_nr}{1-r}, r \neq 1$

Go Online Derive the formula for the sum of a finite geometric series in Expand 5-4.

The sum of a series can be written in shorthand by using **sigma notation**, which uses the Greek uppercase letter S to indicate that you should find a sum.

Key Concept • Sigma Notation		
Symbols	last value of k → $\sum_{k=1}^{n} f(k)$ ← formula for the terms of the series; first value of k →	
Examples	$\sum_{k=1}^{6}(3k+2) = [3(1)+2]+[3(2)+2]+[3(3)+2] + \ldots + [3(6)+2]$ $= 5 + 8 + 11 + \ldots + 20$ $= 75$	Arithmetic series
	$\sum_{k=1}^{8} 5(3)^{k-1} = 5(3)^{1-1} + 5(3)^{2-1} + \ldots + 5(3)^{8-1}$ $= 5(1) + 5(3) + \ldots + 5(2187)$ $= 5 + 15 + \ldots + 10{,}935$ $= 16{,}400$	Geometric series

Example 7 Find the Sum of a Geometric Series

3	Find sums of geometric series	13-29	245-246

Solution

13. **FAMILY** Amanda is researching her ancestry. She records names and birth dates for her parents, their parents, and so on, in an online research tool. If she can locate all of the information, how many names will Amanda record in the generation that is 5 generations before her?

SOLUTION:

Amanda records names and birth dates for her parents, which include 2 people. So, $a_1 = 2$. Each set of parents is 2 people, so $r = 2$.

$a_n = a_1 r^{n-1}$	nth term of a geometric sequence
$a_5 = 2(2)^{5-1}$	$n = 5$, $a_1 = 2$, and $r = 2$
$= 2(2)^4$	Simplify the exponent.
$= 2(16)$	Evaluate the exponent.
$= 32$	Simplify.

In the 5 generation before her, Amanda will record 32 names.

ANSWER:

32

14. **MOORE'S LAW** Gordon Moore, co-founder of Intel, suggested that the number of transistors on a square inch of integrated circuit in a computer chip would double every 18 months. Assuming Moore's law is true, how many times as many transistors would you expect on a square inch of integrated circuit in year 6?

SOLUTION:

The number of transistors on a square inch of integrated circuit in a computer chip would double every 18 months.

18 months is equal to $\frac{18}{12} = 1.5$ years.

After 1.5 years, the number of transistors on a square inch of integrated circuit in a computer chip would be $1 \times 2 = 2$.

After 3 years, the number of transistors on a square inch of integrated circuit in a computer chip would be $2 \times 2 = 4$.

After 4.5 years, the number of transistors on a square inch of integrated circuit in a computer chip would be $4 \times 2 = 8$.

After 6 years, the number of transistors on a square inch of integrated circuit in a computer chip would be $8 \times 2 = 16$.

ANSWER:

16

3	Find sums of geometric series	13-29	245-246	**Solution**

Write an equation for the nth term of each geometric sequence.

15. 3, 9, 27, …

SOLUTION:

Step 1 Find r.

$r = \frac{a_3}{a_2}$ — Divide two consecutive terms.

$= \frac{27}{9}$ or 3 — $a_2 = 9$ and $a_3 = 27$

Step 2 Write the equation.

$a_n = a_1 r^{n-1}$ — nth term of a geometric sequence

$= 3(3)^{n-1}$ — $a_1 = 3$ and $r = 3$

ANSWER:

$a_n = 3(3)^{n-1}$

16. −1, −3, −9, …

SOLUTION:

Step 1 Find r.

$r = \frac{a_3}{a_2}$ — Divide two consecutive terms.

$= \frac{-9}{-3}$ or 3 — $a_2 = -3$ and $a_3 = -9$

Step 2 Write the equation.

$a_n = a_1 r^{n-1}$ — nth term of a geometric sequence

$= -1(3)^{n-1}$ — $a_1 = -1$ and $r = 3$

ANSWER:

$a_n = -1(3)^{n-1}$

17. 2, −6, 18, …

SOLUTION:

Step 1 Find r.

$r = \frac{a_3}{a_2}$ — Divide two consecutive terms.

$= \frac{18}{-6}$ or −3 — $a_2 = -6$ and $a_3 = 18$

Step 2 Write the equation.

$a_n = a_1 r^{n-1}$ — nth term of a geometric sequence

$= 2(-3)^{n-1}$ — $a_1 = 2$ and $r = -3$

ANSWER:

$a_n = 2(-3)^{n-1}$

18. 5, 10, 20, …

SOLUTION:

Step 1 Find r.

$r = \frac{a_3}{a_2}$ — Divide two consecutive terms.

$= \frac{20}{10}$ or 2 — $a_2 = 10$ and $a_3 = 20$

Step 2 Write the equation.

$a_n = a_1 r^{n-1}$ — nth term of a geometric sequence

$= 5(2)^{n-1}$ — $a_1 = 5$ and $r = 2$

ANSWER:

$a_n = 5(2)^{n-1}$

19. $a_4 = 324$ and $r = 3$

SOLUTION:

Step 1 Find a_1.

$a_n = a_1 r^{n-1}$ — nth term of a geometric sequence

$324 = a_1(3)^{4-1}$ — $n = 4$, $a_4 = 324$, and $r = 3$

$324 = a_1(3)^3$ — Simplify the exponent.

$324 = a_1(27)$ — Evaluate the exponent.

$a_1 = 12$ — Solve for a_1.

Step 2 Write the equation.

$a_n = a_1 r^{n-1}$ — nth term of a geometric sequence

$= 12(3)^{n-1}$ — $a_1 = 12$ and $r = 3$

ANSWER:

$a_n = 12(3)^{n-1}$

20. $a_3 = 512$ and $r = \frac{1}{8}$

SOLUTION:

Step 1 Find a_1.

$a_n = a_1 r^{n-1}$ — nth term of a geometric sequence

$512 = a_1\left(\frac{1}{8}\right)^{3-1}$ — $n = 3$, $a_3 = 512$, and $r = \frac{1}{8}$

$512 = a_1\left(\frac{1}{8}\right)^2$ — Simplify the exponent.

$512 = a_1\left(\frac{1}{64}\right)$ — Evaluate the exponent.

$a_1 = 32{,}768$ — Solve for a_1.

Step 2 Write the equation.

$a_n = a_1 r^{n-1}$ — nth term of a geometric sequence

$= 32{,}768\left(\frac{1}{8}\right)^{n-1}$ — $a_1 = 32{,}768$ and $r = \frac{1}{8}$

ANSWER:

$a_n = 32{,}768\left(\frac{1}{8}\right)^{n-1}$

Given a formula for a geometric sequence in recursive or explicit form, translate it to the other form.

21. $a_1 = 3$, $a_n = 0.6a_{n-1}$, and $n \geq 2$

SOLUTION:

Because a_n is defined in terms of the previous term, $a_n = 0.6a_{n-1}$ is a recursive formula of the form $a_n = r \cdot a_{n-1}$. Thus $r = 0.6$. Now, write the explicit formula.

$$\begin{aligned} a_n &= a_1 r^{n-1} && \text{Explicit formula for a geometric sequence} \\ &= 3(0.6)^{n-1} && a_1 = 3 \text{ and } r = 0.6 \end{aligned}$$

The explicit formula for $a_1 = 3$, $a_n = 0.6a_{n-1}$, and $n \geq 2$ is $a_n = 3(0.6)^{n-1}$.

ANSWER:

$a_n = 3(0.6)^{n-1}$

22. $a_n = 0.8(2)^{n-1}$

SOLUTION:

Because a_n is defined in terms of n, $a_n = 0.8(2)^{n-1}$ is an explicit formula of the form $a_n = a_1 r^{n-1}$. Thus $a_1 = 0.8$ and $r = 2$. Now, write the recursive formula.

$$\begin{aligned} a_n &= r a_{n-1} && \text{Recursive formula for a geometric sequence} \\ &= 2a_{n-1} && r = 2 \end{aligned}$$

The explicit formula for $a_n = 0.8(2)^{n-1}$ is $a_1 = 0.8$, $a_n = 2a_{n-1}$, and $n \geq 2$.

ANSWER:

$a_1 = 0.8$, $a_n = 2a_{n-1}$, and $n \geq 2$

23. $a_1 = -1$, $a_n = \frac{1}{2}a_{n-1}$, and $n \geq 2$

SOLUTION:

Because a_n is defined in terms of the previous term, $a_n = \frac{1}{2}a_{n-1}$ is a recursive formula of the form $a_n = r \cdot a_{n-1}$. Thus $r = \frac{1}{2}$. Now, write the explicit formula.

$$\begin{aligned} a_n &= a_1 r^{n-1} && \text{Explicit formula for a geometric sequence} \\ &= -1\left(\frac{1}{2}\right)^{n-1} \text{ or } -\left(\frac{1}{2}\right)^{n-1} && a_1 = -1 \text{ and } r = \frac{1}{2} \end{aligned}$$

The explicit formula for $a_1 = -1$, $a_n = \frac{1}{2}a_{n-1}$, and $n \geq 2$ is $a_n = -\left(\frac{1}{2}\right)^{n-1}$.

ANSWER:

$a_n = -\left(\frac{1}{2}\right)^{n-1}$

24. $a_n = -\frac{2}{3}(6)^{n-1}$

SOLUTION:

Because a_n is defined in terms of n, $a_n = -\frac{2}{3}(6)^{n-1}$ is an explicit formula of the form $a_n = a_1 r^{n-1}$. Thus $a_1 = -\frac{2}{3}$ and $r = 6$. Now, write the recursive formula.

$a_n = ra_{n-1}$ Recursive formula for a geometric sequence
$= 6a_{n-1}$ $r = 6$

The explicit formula for $a_n = -\frac{2}{3}(6)^{n-1}$ is $a_1 = -\frac{2}{3}$, $a_n = 6a_{n-1}$, and $n \geq 2$.

ANSWER:

$a_1 = -\frac{2}{3}$, $a_n = 6a_{n-1}$, and $n \geq 2$

Find the geometric means of each sequence.

25. 4, ? , ? , ? , 64

SOLUTION:

Step 1 Find the total number of terms.

Because there are three terms between the first and : 2 = 5 total terms, so $n = 5$.

Step 2 Find r.

$a_n = a_1 r^{n-1}$ — nth term of a geometric sequence
$64 = 4r^{5-1}$ — $n = 5$, $a_5 = 64$, and $a_1 = 4$
$16 = r^4$ — Divide each side by 4.
$r = \pm 2$ — Take the 4th root of each side.

Step 3 Use $r = 2$ to find three geometric means.

4 → 8 → 16 → 3
× 2, × 2, × 2

The geometric means are 8, 16, and 32.

Use $r = -2$ to find three geometric means.

4 → –8 → 16 → –
× –2, × –2, × –2

The geometric means are –8, 16, and –32.

ANSWER:

±8, 16, ±32

26. 1, ? , ? , ? , 81

SOLUTION:

Step 1 Find the total number of terms.

Because there are three terms between the first and : 2 = 5 total terms, so $n = 5$.

Step 2 Find r.

$a_n = a_1 r^{n-1}$ — nth term of a geometric sequence
$81 = 1r^{5-1}$ — $n = 5$, $a_5 = 81$, and $a_1 = 1$
$81 = r^4$ — Simplify.
$r = \pm 3$ — Take the 4th root of each side.

Step 3 Use $r = 3$ to find three geometric means.

1 → 3 → 9 → 2
× 3, × 3, × 3

The geometric means are 3, 9, and 27.

Use $r = -3$ to find three geometric means.

1 → –3 → 9 → –
× –3, × –3, × –3

The geometric means are –3, 9, and –27.

ANSWER:

±3, 9, ±27

27. 38; 228; ? ; 8208; 49,248; ...

SOLUTION:

Step 1 Find the total number of terms.

Because there are three terms between the first and 2 = 5 total terms, so $n = 5$.

Step 2 Find r.

$a_n = a_1 r^{n-1}$ — nth term of a geometric sequence
$49{,}248 = 38r^{5-1}$ — $n = 5$, $a_5 = 49{,}248$, and $a_1 = 38$
$1296 = r^4$ — Divide each side by 38.
$r = \pm 6$ — Take the 4th root of each side.

Step 3 The first, second, fourth, and fifth terms are to find the geometric mean.

38 → 228 → 1368 → 82
× 6, × 6, × 6

The geometric mean is 1368.

ANSWER:

1368

28. 51; ? ; 4131; ? ; 334,611; …

SOLUTION:

Step 1 Find the total number of terms.

Because there are three terms between the first and l
2 = 5 total terms, so $n = 5$.

Step 2 Find r.

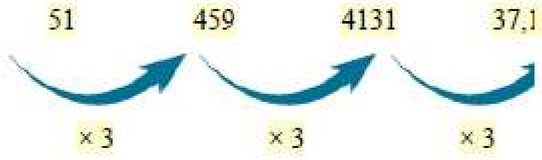
$a_n = a_1 r^{n-1}$ — nth term of a geometric sequence

$334{,}611 = 51r^{5-1}$ — $n = 5$, $a_5 = 334{,}611$, and $a_1 = 51$

$6561 = r^4$ — Divide each side by 51.

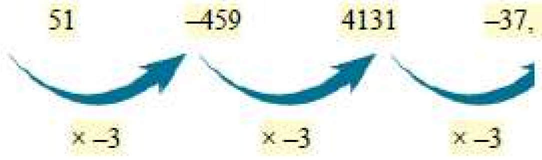
$r = \pm 9$ — Take the 4th root of each side.

Step 3 Use $r = 9$ to find three geometric means.



The geometric means are 459 and 37,179.

Use $r = -9$ to find three geometric means.



The geometric means are –459 and –37,179.

ANSWER:

±459; ±37,179

29. **SCIENTIFIC RESEARCH** Scientific balloons ca equipment to observe or conduct experiments. The l Balloon Program generally tries to fly balloons abov 90,000 feet. Suppose a balloon rises 1000 feet in the minute after it is launched. For the next hour, each mi 1% more than it rose in the previous minute.

a. Complete the table to show the height of the ballo various times after launch.

Time (min)	1	2	3	4	5
Height (ft)					

b. After an hour will the balloon have reached its targ 80,000 – 90,000 feet? Explain.

SOLUTION:

a. A balloon rises 1000 feet in the first minute after it launched. So, $a_1 = 1000$ and $r = 1.01$.

$n = 1$:

$S_n = \frac{a_1 - a_1 r^n}{1 - r}$ — Sum formula

$S_n = \frac{1000 - 1000(1.01)^1}{1 - 1.01}$ — $a_1 = 1000$, $r = 1.01$, and $n = 1$

$S_n = \frac{-10}{-0.01}$ — Simplify the numerator and denominator.

$S_n = 1000$ — Divide.

$n = 2$:

$S_n = \frac{a_1 - a_1 r^n}{1 - r}$ — Sum formula

$S_n = \frac{1000 - 1000(1.01)^2}{1 - 1.01}$ — $a_1 = 1000$, $r = 1.01$, and $n = 2$

$S_n = \frac{-20.1}{-0.01}$ — Simplify the numerator and denominator.

$S_n = 2010$ — Divide.

$n = 3$:

$S_n = \frac{a_1 - a_1 r^n}{1 - r}$ — Sum formula

$S_n = \frac{1000 - 1000(1.01)^3}{1 - 1.01}$ — $a_1 = 1000$, $r = 1.01$, and $n = 3$

$S_n = \frac{-30.301}{-0.01}$ — Simplify the numerator and denominator.

$S_n = 3030.1$ — Divide.

$n = 4$:

$S_n = \frac{a_1 - a_1 r^n}{1 - r}$ — Sum formula

$S_n = \frac{1000 - 1000(1.01)^4}{1 - 1.01}$ — $a_1 = 1000$, $r = 1.01$, and $n = 4$

$S_n = \frac{-40.60401}{-0.01}$ — Simplify the numerator and denominator.

$S_n = 4060.401$ — Divide.

$n = 5$:

$S_n = \frac{a_1 - a_1 r^n}{1 - r}$ — Sum formula

$S_n = \frac{1000 - 1000(1.01)^5}{1 - 1.01}$ — $a_1 = 1000$, $r = 1.01$, and $n = 5$

$S_n = \frac{-51.0100501}{-0.01}$ — Simplify the numerator and denominator.

$S_n = 5101.00501$ — Divide.

$n = 6$:

$S_n = \frac{a_1 - a_1 r^n}{1 - r}$ — Sum formula

$S_n = \frac{1000 - 1000(1.01)^6}{1 - 1.01}$ — $a_1 = 1000$, $r = 1.01$, and $n = 6$

$S_n = \frac{-61.5201506}{-0.01}$ — Simplify the numerator and denominator.

$S_n = 6152.01506$ — Divide.

b. 1 hour is equal to 60 minutes, so $n = 60$:

$S_n = \frac{a_1 - a_1 r^n}{1 - r}$ — Sum formula

$S_n = \frac{1000 - 1000(1.01)^{60}}{1 - 1.01}$ — $a_1 = 1000$, $r = 1.01$, and $n = 60$

$S_n = \frac{-816.697}{-0.01}$ — Simplify the numerator and denominator.

$S_n = 81{,}669.7$ — Divide.

The height after 60 minutes is about 81,669.7 ft. So, will reach its target height of 80,000 – 90,000 feet af

ANSWER:

a.

Time (s)	1	2	3	4	5
Height (ft)	1000	2010	3030.1	4060.401	5101.00501

b. Yes; the height after 60 minutes is about 81,669.7

4	Simplify rational expressions by multiplying and dividing	24-35	316

Simplify each expression.

24. $\frac{y^2 + 8y + 15}{y - 6} \cdot \frac{y^2 - 9y + 18}{y^2 - 9}$

25. $\frac{c^2 - 6c - 16}{c^2 - d^2} \div \frac{c^2 - 8c}{c + d}$

26. $\frac{x^2 + 9x + 20}{8x + 16} \cdot \frac{4x^2 + 16x + 16}{x^2 - 25}$

4	Simplify rational expressions by multiplying and dividing	24-35	316

27. $\dfrac{3a^2 + 6a + 3}{a^2 - 3a - 10} \div \dfrac{12a^2 - 12}{a^2 - 4}$

28. $\dfrac{9 - x^2}{x^2 - 4x - 21} \cdot \left(\dfrac{2x^2 + 7x + 3}{2x^2 - 15x + 7}\right)^{-1}$

29. $\left(\dfrac{2x^2 + 2x - 12}{x^2 + 4x - 5}\right)^{-1} \cdot \dfrac{2x^3 - 8x}{x^2 - 2x - 35}$

4	Simplify rational expressions by multiplying and dividing	24-35	316

30. $\left(\frac{3xy^3z}{2a^2bc^2}\right)^3 \cdot \frac{16a^4b^3c^5}{15x^7yz^3}$

31. $\frac{20x^2y^6z^{-2}}{3a^3c^2} \cdot \left(\frac{16x^3y^3}{9acz}\right)^{-1}$

32. $\frac{\frac{8x^2 - 10x - 3}{10x^2 + 35x - 20}}{\frac{2x^2 + x - 6}{4x^2 + 18x + 8}}$

4	Simplify rational expressions by multiplying and dividing	24-35	316

33. $\dfrac{\dfrac{2x^2 + 7x - 30}{-6x^2 + 13x + 5}}{\dfrac{4x^2 + 12x - 72}{3x^2 - 11x - 4}}$

34. $\dfrac{x^2 + 4x - 32}{2x^2 + 9x - 5} \cdot \dfrac{3x^2 - 75}{3x^2 - 11x - 4} \div \dfrac{6x^2 - 18x - 60}{x^3 - 4x}$

35. $\dfrac{8x^2 + 10x - 3}{3x^2 - 12x - 36} \div \dfrac{2x^2 - 5x - 12}{3x^2 - 17x - 6} \cdot \dfrac{4x^2 + 3x - 1}{4x^2 - 40x + 24}$

4	Simplify rational expressions by multiplying and dividing	24-35	316

solution method

Lesson 7-1

Multiplying and Dividing Rational Expressions

Learn Simplifying Rational Expressions

A **rational expression** is a ratio of two polynomial expressions.

Because variables in algebra often represent real numbers, operations with rational numbers and rational expressions are similar. For example, when you write a fraction in simplest form, you divide the numerator and denominator by the greatest common factor (GCF).

$$\frac{35}{40} = \frac{\cancel{5} \cdot 7}{\cancel{5} \cdot 8} = \frac{7}{8} \qquad \text{GCF} = 5$$

You use the same process to simplify a rational expression.

$$\frac{x^2 + 7x + 10}{x^2 - x - 6} = \frac{(x+5)\cancel{(x+2)}}{(x-3)\cancel{(x+2)}} = \frac{(x+5)}{(x-3)} \qquad \text{GCF} = x + 2$$

Sometimes, you can also factor out −1 in the numerator or denominator to help simplify a rational expression.

Example 2 Simplify by Using −1

Learn Multiplying and Dividing Rational Expressions

The method for multiplying and dividing fractions also works with rational expressions.

Key Concept • Multiplying Rational Expressions

Words: To multiply rational expressions, multiply the numerators and the denominators.

Symbols: For all rational expressions $\frac{a}{b}$ and $\frac{c}{d}$ with $b \neq 0$ and $d \neq 0$, $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

Key Concept • Dividing Rational Expressions

Words: To divide rational expressions, multiply the dividend by the reciprocal of the divisor.

Symbols: For all rational expressions $\frac{a}{b}$ and $\frac{c}{d}$ with $b \neq 0$, $c \neq 0$, and $d \neq 0$, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$.

A **complex fraction** is a rational expression with a numerator and/or denominator that is also a rational expression. To simplify a complex fraction, first rewrite it as a division expression.

Example 3 Multiply and Divide Rational Expressions

Example 5 Simplify Complex Fractions

24. $\frac{y^2+8y+15}{y-6} \cdot \frac{y^2-9y+18}{y^2-9}$

SOLUTION:

$\frac{y^2+8y+15}{y-6} \cdot \frac{y^2-9y+18}{y^2-9} = \frac{(y+5)(y+3)}{(y-6)} \cdot \frac{(y-6)(y-3)}{(y+3)(y-3)}$ Factor and eliminate factors.

$= y+5$ Multiply by the reciprocal.

ANSWER:

$y+5$

25. $\frac{c^2-6c-16}{c^2-d^2} \div \frac{c^2-8c}{c+d}$

SOLUTION:

$\frac{c^2-6c-16}{c^2-d^2} \div \frac{c^2-8c}{c+d} = \frac{c^2-6c-16}{c^2-d^2} \cdot \frac{c+d}{c^2-8c}$ Multiply by the reciprocal.

$= \frac{c^2-6c-16}{c^2-d^2} \cdot \frac{c+d}{c(c-8)}$ Distributive Property

$= \frac{(c-8)(c+2)}{(c+d)(c-d)} \cdot \frac{(c+d)}{c(c-8)}$ Factor and eliminate factors.

$= \frac{(c+2)}{c(c-d)}$ Simplify.

ANSWER:

$\frac{(c+2)}{c(c-d)}$

26. $\frac{x^2+9x+20}{8x+16} \cdot \frac{4x^2+16x+16}{x^2-25}$

SOLUTION:

$\frac{x^2+9x+20}{8x+16} \cdot \frac{4x^2+16x+16}{x^2-25} = \frac{x^2+9x+20}{8(x+2)} \cdot \frac{4(x^2+4x+4)}{x^2-25}$ Distributive Property

$= \frac{(x+4)(x+5)}{8(x+2)} \cdot \frac{4(x+2)(x+2)}{(x+5)(x-5)}$ Factor and eliminate factors.

$= \frac{(x+4)(x+2)}{2(x-5)}$ Multiply and simplify.

ANSWER:

$\frac{(x+4)(x+2)}{2(x-5)}$

27. $\frac{3a^2+6a+3}{a^2-3a-10} \div \frac{12a^2-12}{a^2-4}$

SOLUTION:

$\frac{3a^2+6a+3}{a^2-3a-10} \div \frac{12a^2-12}{a^2-4} = \frac{3a^2+6a+3}{a^2-3a-10} \cdot \frac{a^2-4}{12a^2-12}$ Multiply by the reciprocal.

$= \frac{3(a^2+2a+1)}{a^2-3a-10} \cdot \frac{a^2-4}{12(a^2-1)}$ Distributive Property

$= \frac{3(a+1)(a+1)}{(a+2)(a-5)} \cdot \frac{(a+2)(a-2)}{12(a+1)(a-1)}$ Factor and eliminate common factors.

$= \frac{(a+1)(a-2)}{4(a-5)(a-1)}$ Simplify.

ANSWER:

$\frac{(a+1)(a-2)}{4(a-5)(a-1)}$

28. $\frac{9-x^2}{x^2-4x-21} \cdot \left(\frac{2x^2+7x+3}{2x^2-15x+7}\right)^{-1}$

SOLUTION:

$\frac{9-x^2}{x^2-4x-21} \cdot \left(\frac{2x^2+7x+3}{2x^2-15x+7}\right)^{-1} = \frac{9-x^2}{x^2-4x-21} \cdot \frac{2x^2-15x+7}{2x^2+7x+3}$ Multiply by the reciprocal.

$= \frac{(3+x)(3-x)}{(x-7)(x+3)} \cdot \frac{(2x-1)(x-7)}{(2x+1)(x+3)}$ Factor and eliminate common factors.

$= \frac{(3-x)(2x-1)}{(x+3)(2x+1)}$ Simplify.

ANSWER:

$\frac{(3-x)(2x-1)}{(x+3)(2x+1)}$

29. $\left(\frac{2x^2+2x-12}{x^2+4x-5}\right)^{-1} \cdot \frac{2x^3-8x}{x^2-2x-35}$

SOLUTION:

$\left(\frac{2x^2+2x-12}{x^2+4x-5}\right)^{-1} \cdot \frac{2x^3-8x}{x^2-2x-35} = \frac{x^2+4x-5}{2x^2+2x-12} \cdot \frac{2x^3-8x}{x^2-2x-35}$ Multiply by the reciprocal.

$= \frac{x^2+4x-5}{2(x^2+x-6)} \cdot \frac{2x(x^2-4)}{x^2-2x-35}$ Distributive Property

$= \frac{(x+5)(x-1)}{2(x+3)(x-2)} \cdot \frac{2x(x+2)(x-2)}{(x+5)(x-7)}$ Factor and eliminate common factors.

$= \frac{x(x+2)(x-1)}{(x+3)(x-7)}$ Simplify.

ANSWER:

$\frac{x(x+2)(x-1)}{(x+3)(x-7)}$

30. $\left(\frac{3xy^3z}{2a^2bc^2}\right)^3 \cdot \frac{16a^4b^3c^5}{15x^7yz^3}$

SOLUTION:

$\left(\frac{3xy^3z}{2a^2bc^2}\right)^3 \cdot \frac{16a^4b^3c^5}{15x^7yz^3} = \frac{27x^3y^9z^3}{8a^6b^3c^6} \cdot \frac{16a^4b^3c^5}{15x^7yz^3}$ $(a^m)^n = a^{m \cdot n}$

$= \frac{(3+x)(3-x)}{(x-7)(x+3)} \cdot \frac{(2x-1)(x-7)}{(2x+1)(x+3)}$ Distributive Property

$= \frac{(3-x)(2x-1)}{(x+3)(2x+1)}$ Simplify.

[illegible]

ANSWER:

$\frac{18y^8}{5a^2cx^4}$

31. $\left(\frac{3xy^3z}{2a^2bc^2}\right)^3 \cdot \frac{16a^4b^3c^5}{15x^7yz^3}$

SOLUTION:

$\left(\frac{3xy^3z}{2a^2bc^2}\right)^3 \cdot \frac{16a^4b^3c^5}{15x^7yz^3} = \frac{27x^3y^9z^3}{8a^6b^3c^6} \cdot \frac{16a^4b^3c^5}{15x^7yz^3}$ $(a^m)^n = a^{m \cdot n}$

$= \frac{(3+x)(3-x)}{(x-7)(x+3)} \cdot \frac{(2x-1)(x-7)}{(2x+1)(x+3)}$ Distributive Property

$= \frac{(3-x)(2x-1)}{(x+3)(2x+1)}$ Simplify.

[illegible]

ANSWER:

$\frac{18y^8}{5a^2cx^4}$

4	Simplify rational expressions by multiplying and dividing	24-35	316

32. $\dfrac{\dfrac{8x^2-10x-3}{10x^2+35x-20}}{\dfrac{2x^2+x-6}{4x^2+18x+8}}$

SOLUTION:

$\dfrac{\dfrac{8x^2-10x-3}{10x^2+35x-20}}{\dfrac{2x^2+x-6}{4x^2+18x+8}} = \dfrac{8x^2-10x-3}{10x^2+35x-20} \div \dfrac{2x^2+x-6}{4x^2+18x+8}$ Express as a division expression.

$= \dfrac{8x^2-10x-3}{10x^2+35x-20} \cdot \dfrac{4x^2+18x+8}{2x^2+x-6}$ Multiply by the reciprocal.

$= \dfrac{8x^2-10x-3}{5(2x^2+7x-4)} \cdot \dfrac{2(2x^2+9x+4)}{2x^2+x-6}$ Distributive Property.

$= \dfrac{(2x-3)(4x+1)}{5(2x-1)(x+4)} \cdot \dfrac{2(x+4)(2x+1)}{(x+2)(2x-3)}$ Factor and eliminate factors.

$= \dfrac{2(4x+1)(2x+1)}{5(2x-1)(x+2)}$ Simplify.

ANSWER:

$\dfrac{2(4x+1)(2x+1)}{5(2x-1)(x+2)}$

33. $\dfrac{\dfrac{2x^2+7x-30}{-6x^2+13x+5}}{\dfrac{4x^2+12x-72}{3x^2-11x-4}}$

SOLUTION:

$\dfrac{\dfrac{2x^2+7x-30}{-6x^2+13x+5}}{\dfrac{4x^2+12x-72}{3x^2-11x-4}} = \dfrac{2x^2+7x-30}{-6x^2+13x+5} \div \dfrac{4x^2+12x-72}{3x^2-11x-4}$ Express as a division expression.

$= \dfrac{2x^2+7x-30}{-6x^2+13x+5} \cdot \dfrac{3x^2-11x-4}{4x^2+12x-72}$ Multiply by the reciprocal.

$= \dfrac{2x^2+7x-30}{-1(6x^2-13x-5)} \cdot \dfrac{3x^2-11x-4}{4(x^2+3x-18)}$ Distributive Property.

$= \dfrac{(x+6)(2x-5)}{-1(2x-5)(3x+1)} \cdot \dfrac{(3x+1)(x-4)}{4(x-3)(x+6)}$ Factor and eliminate factors.

$= \dfrac{(x-4)}{-4(x-3)}$ Simplify.

ANSWER:

$\dfrac{x-4}{-4(x-3)}$

35. $\dfrac{8x^2+10x-3}{3x^2-12x-36} \div \dfrac{2x^2-5x-12}{3x^2-17x-6} \cdot \dfrac{4x^2+3x-1}{4x^2-40x+24}$

SOLUTION:

$\dfrac{8x^2+10x-3}{3x^2-12x-36} \div \dfrac{2x^2-5x-12}{3x^2-17x-6} \cdot \dfrac{4x^2+3x-1}{4x^2-40x+24} = \dfrac{8x^2+10x-3}{3x^2-12x-36} \cdot \dfrac{3x^2-17x-6}{2x^2-5x-12} \cdot \dfrac{4x^2+3x-1}{4x^2-40x+24}$ Multiply by the reciprocal.

[illegible] Distributive Property.

[illegible] Factor and eliminate factors.

[illegible] Simplify.

ANSWER:

$\dfrac{(4x-1)^2(3x+1)(x+1)}{12(x+2)(x-4)(x^2-10x+6)}$

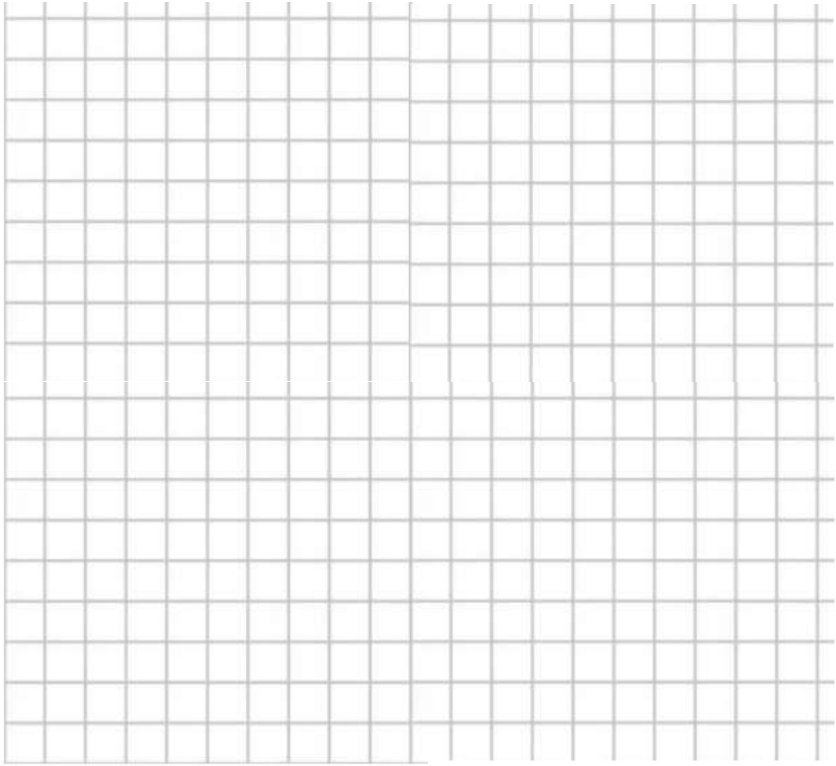
5	Graph rational functions with vertical and horizontal asymptotes	1-10; Example 1 & 3	343; 337-338-339-340

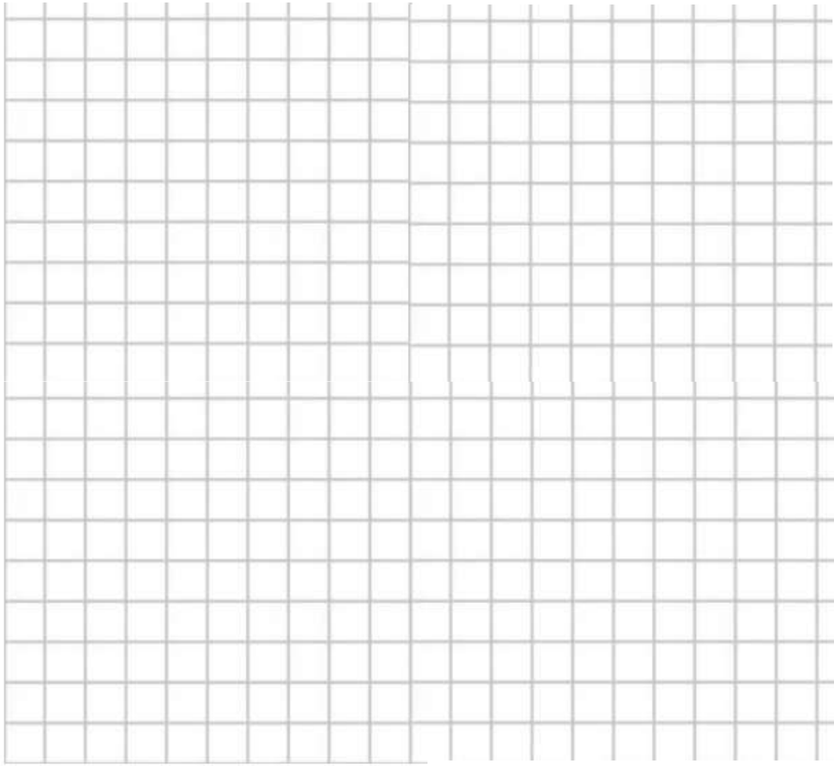
Example 1

Graph each function.

1. $f(x) = \frac{x^4}{6x + 12}$

2. $f(x) = \frac{x^3}{8x - 4}$





5	Graph rational functions with vertical and horizontal asymptotes	1-10; Example 1 & 3	343; 337-338-339-340

3. $f(x) = \dfrac{x^4 - 16}{x^2 - 1}$

4. $f(x) = \dfrac{x^3 + 64}{16x - 24}$

5	Graph rational functions with vertical and horizontal asymptotes	1-10; Example 1 & 3	343; 337-338-339-340

Example 2

5. INTERNET An Internet service provider charges customers a $60 installation fee plus $30 per month for Internet service. A function that models the average monthly cost is $f(x) = \frac{60 + 30x}{x}$, where x is the number of months.

a. Graph the function.

b. Find the x- and y-intercepts and end behavior of the graph.

c. Find the average monthly cost to a customer that has Internet service for 8 months.

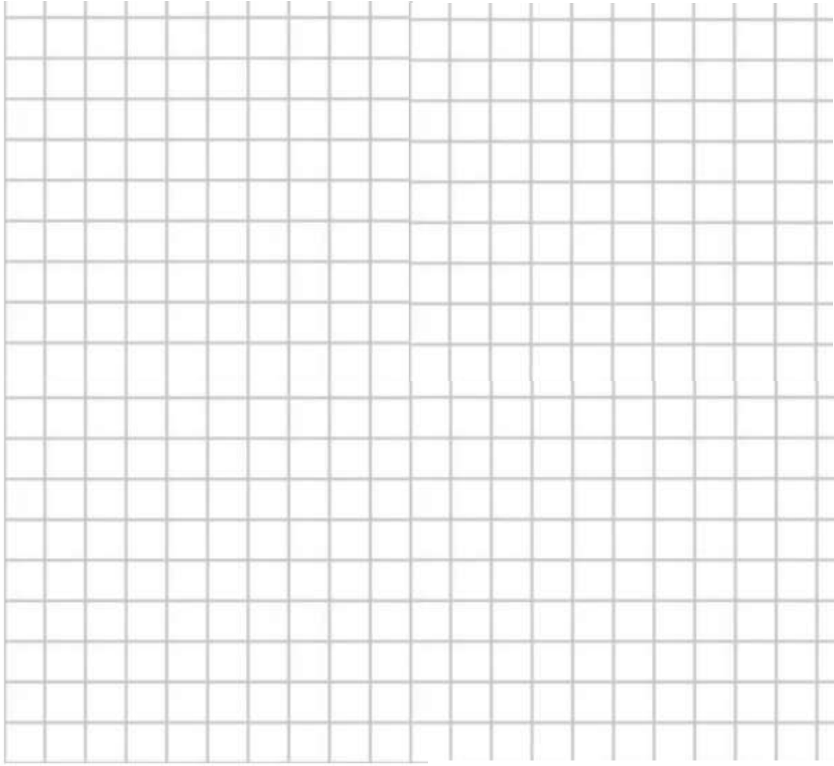
5	Graph rational functions with vertical and horizontal asymptotes	1-10; Example 1 & 3	343; 337-338-339-340

6. SALES The quantity of a certain product sold in week x is approximated by the function $f(x) = \frac{80x}{x^2 + 40}$.

a. Graph the function.

b. Find the x- and y-intercepts and the end behavior of the graph.

c. During which week(s) did 5 of the products sell?



5	Graph rational functions with vertical and horizontal asymptotes	1-10; Example 1 & 3	343; 337-338-339-340

7. FACTORY The cost in cents to create a certain part of a small engine is modeled by $f(x) = \frac{18x - 12}{6x}$, where x is the number of parts made.

a. Graph the function.

b. Find the x- and y-intercepts and the end behavior of the graph.

c. About how much does the 6th part cost to make?

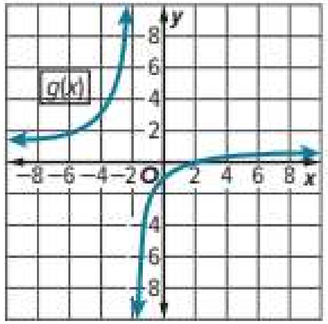
5	Graph rational functions with vertical and horizontal asymptotes	1-10; Example 1 & 3	343; 337-338-339-340

Example 3

For Exercises 8-10, consider the given function and the function shown in the graph.

a. Copy the graph. Graph the given function.
b. Which function has the greater *y*-intercept?
c. Compare the asymptotes of the two functions.

8. $f(x) = \frac{x - 5}{3x + 5}$ and $g(x)$ shown in the graph



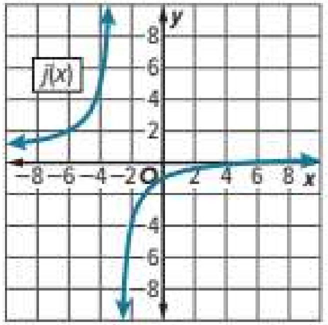
5	Graph rational functions with vertical and horizontal asymptotes	1-10; Example 1 & 3	343; 337-338-339-340

Example 3

For Exercises 8-10, consider the given function and the function shown in the graph.

a. Copy the graph. Graph the given function.
b. Which function has the greater *y*-intercept?
c. Compare the asymptotes of the two functions.

9. $h(x) = \frac{x+1}{4x-4}$ and $j(x)$ shown in the graph



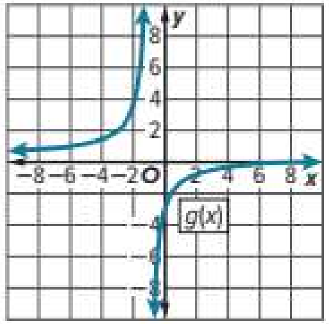
5	Graph rational functions with vertical and horizontal asymptotes	1-10; Example 1 & 3	343; 337-338-339-340

Example 3

For Exercises 8-10, consider the given function and the function shown in the graph.

a. Copy the graph. Graph the given function.
b. Which function has the greater *y*-intercept?
c. Compare the asymptotes of the two functions.

10. $f(x) = \frac{x-3}{2x+7}$ and $g(x)$ shown in the graph



5	Graph rational functions with vertical and horizontal asymptotes	1-10; Example 1 & 3	343; 337-338-339-340	**Solution**

solution method

Lesson 7-4

Graphing Rational Functions

Learn Graphing Rational Functions with Vertical and Horizontal Asymptotes

A **rational function** has an equation of the form $f(x) = \frac{a(x)}{b(x)}$, where $a(x)$ and $b(x)$ are polynomial functions and $b(x) \neq 0$.

Key Concept • Vertical and Horizontal Asymptotes

If $f(x) = \frac{a(x)}{b(x)}$, $a(x)$ and $b(x)$ are polynomial functions with no common factors other than 1, and $b(x) \neq 0$, then:

- $f(x)$ has a vertical asymptote whenever $b(x) = 0$.
- $f(x)$ has at most one horizontal asymptote.
  - If the degree of $a(x)$ is greater than the degree of $b(x)$, there is no horizontal asymptote.
  - If the degree of $a(x)$ is less than the degree of $b(x)$, the horizontal asymptote is the line $y = 0$.
  - If the degree of $a(x)$ equals the degree of $b(x)$, the horizontal asymptote is the line $y = \frac{\text{leading coefficient of } a(x)}{\text{leading coefficient of } b(x)}$.

Example 1 Graph with No Horizontal Asymptotes

Example 2 Use Graphs of Rational Functions

Example 3 Compare Rational Functions

Graph each function.

1. $f(x) = \frac{x^4}{6x+12}$

SOLUTION:

Step 1 Find the zeros.

Set $a(x) = 0$

$x^4 = 0$

$x = 0$

There is a zero at $x = 0$.

Step 2 Find the asymptotes.

Find the vertical asymptote. Set $b(x) = 0$.

$6x + 12 = 0$

$6x = -12$

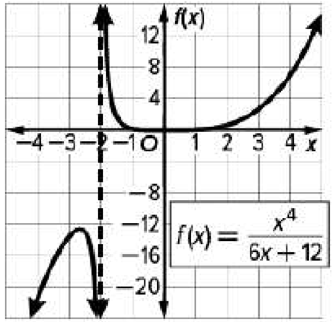
$x = -2$

Because the degree of the numerator is greater than the degree of the denominator, there is no horizontal asymptote.

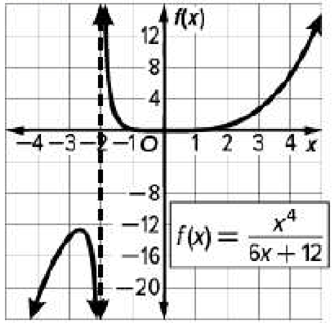
Step 3 Draw the graph.

Graph the asymptote. Then make a table of values, and graph.

x	$f(x)$
–4	$-21\frac{1}{3}$
–3	–13.5
–2	undefined
–1	$\frac{1}{6}$
0	0
1	$\frac{1}{18}$
2	$\frac{2}{3}$
3	$2\frac{7}{10}$
4	$7\frac{1}{9}$



ANSWER:



2. $f(x) = \frac{x^3}{8x-4}$

SOLUTION:

Step 1 Find the zeros.

Set $a(x) = 0$

$x^3 = 0$

$x = 0$

There is a zero at $x = 0$.

Step 2 Find the asymptotes.

Find the vertical asymptote. Set $b(x) = 0$.

$8x - 4 = 0$

$8x = 4$

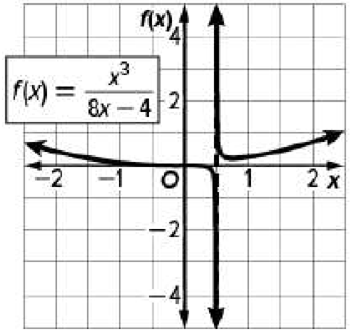
$x = \frac{1}{2}$

Because the degree of the numerator is greater than the degree of the denominator, there is no horizontal asymptote.

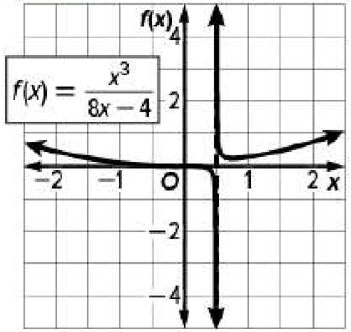
Step 3 Draw the graph.

Graph the asymptote. Then make a table of values, and graph.

x	$f(x)$
−4	$1\frac{7}{9}$
−3	$\frac{27}{28}$
−2	$\frac{2}{5}$
−1	$\frac{1}{12}$
0	0
1	$\frac{1}{4}$
2	$\frac{2}{3}$
3	$1\frac{7}{20}$
4	$2\frac{2}{7}$



ANSWER:



3. $f(x) = \frac{x^4 - 16}{x^2 - 1}$

SOLUTION:

Step 1 Find the zeros.

Set $a(x) = 0$

$x^4 - 16 = 0$

$x^4 = 16$

$x = \pm 2$

There are zeros at $x = -2$ and 2.

Step 2 Find the asymptotes.

Find the vertical asymptote. Set $b(x) = 0$.

$x^2 - 1 = 0$

$x^2 = 1$

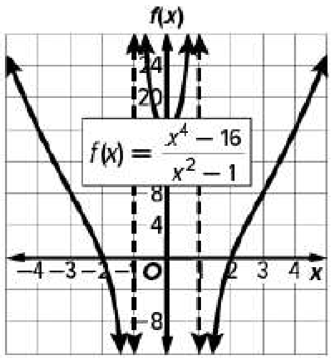
$x = \pm 1$

Because the degree of the numerator is greater than the degree of the denominator, there is no horizontal asymptote.

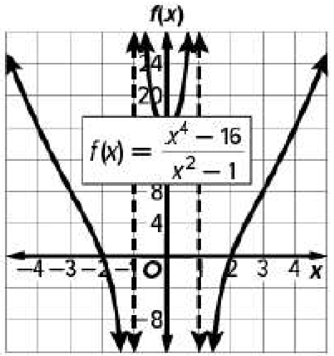
Step 3 Draw the graph.

Graph the asymptote. Then make a table of values, and graph.

x	$f(x)$
−4	16
−3	$8\frac{1}{8}$
−2	0
−1	undefined
0	0
1	undefined
2	16
3	$8\frac{1}{8}$
4	16



ANSWER:



4. $f(x) = \frac{x^3 + 64}{16x - 24}$

SOLUTION:

Step 1 Find the zeros.

Set $a(x) = 0$

$x^3 + 64 = 0$

$x^3 = -64$

$x = -4$

There is a zero at $x = -4$.

Step 2 Find the asymptotes.

Find the vertical asymptote. Set $b(x) = 0$.

$16x - 24 = 0$

$16x = 24$

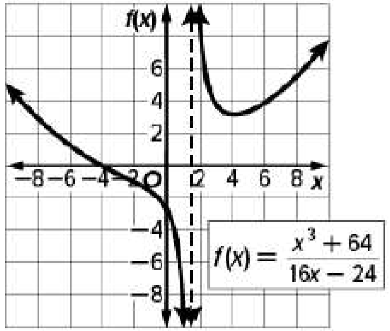
$x = 1\frac{1}{2}$

Because the degree of the numerator is greater than the degree of the denominator, there is no horizontal asymptote.

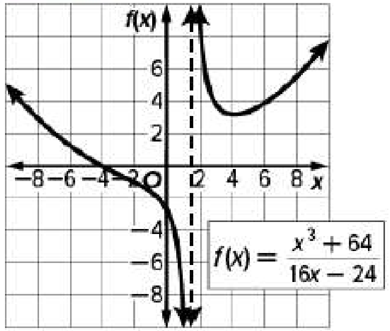
Step 3 Draw the graph.

Graph the asymptote. Then make a table of values, and graph.

x	$f(x)$
–4	0
–3	$1\frac{13}{24}$
–2	–1
–1	$-1\frac{23}{40}$
0	$-2\frac{2}{3}$
1	$-8\frac{1}{8}$
2	9
3	$3\frac{19}{24}$
4	$3\frac{1}{5}$



ANSWER:



5. **INTERNET** An Internet service provider charges customers a \$60 installation fee plus \$30 per month for Internet service. A function that models the average monthly cost is $f(x) = \frac{60 + 30x}{x}$, where x is the number of months.

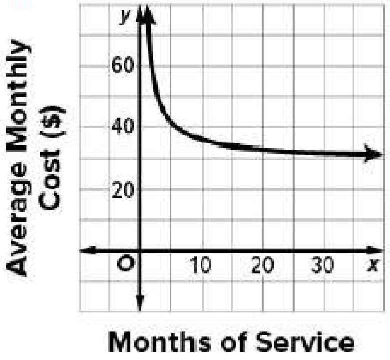
a. Graph the function.

b. Find the x-and y- intercepts and end behavior of the graph.

c. Find the average monthly cost to a customer that has Internet service for 8 months.

SOLUTION:

a. Because the degree of $a(x)$ = the degree of $b(x)$, the horizontal asymptote is the line $y = \frac{\text{leading coefficient of } a(x)}{\text{leading coefficient of } b(x)}$, so $y = \frac{30}{1}$ or $y = 30$.



b. Find the x-intercept by setting $y = 0$.

$60 + 30x = 0$

$30x = -60$

$x = -2$

The x-intercept is –2.

To find the y-intercept, setting $x = 0$ is undefined, so there is no y-intercept.

The end behavior is: as $x \to -\infty$, $f(x) \to 30$ and as $x \to \infty$, $f(x) \to 30$.

c. To find the average monthly cost for 8 months, substitute 8 for x into the function.

$f(x) = \frac{60 + 30x}{x}$

$= \frac{60 + 30(8)}{8}$

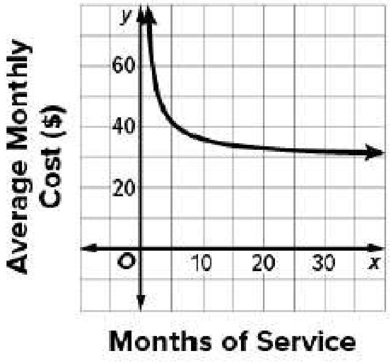
$= \frac{300}{8}$

$= 37.5$

The average monthly cost for 8 months is \$37.50.

ANSWER:

a.



b. x-intercept: –2; y-intercept: none; end behavior: As $x \to -\infty$, $f(x) \to 30$ and as $x \to \infty$, $f(x) \to 30$.

c. \$37.50

6. **SALES** The quantity of a certain product sold in week x is approximated by the function $f(x) = \frac{80x}{x^2 + 40}$.

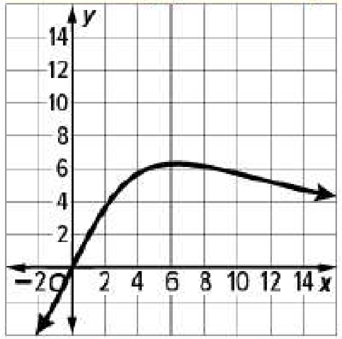
a. Graph the function.

b. Find the x-and y-intercepts and end behavior of the graph.

c. During which week(s) did 5 of the products sell?

SOLUTION:

a. Because the degree of $a(x) <$ the degree of $b(x)$, the horizontal asymptote is the line $y = 0$.



To find the vertical asymptote, set $b(x) = 0$.

$x^2 + 40 = 0$

$x^2 = -40$

$x = \sqrt{-40}$

This is not a real number, so there is no vertical asymptote.

b. Find the x-intercept by setting $y = 0$.

$80x = 0$

$x = 0$

The x-intercept is 0.
Find the y-intercept by setting $x = 0$.

$y = \frac{80x}{x^2 + 40}$

$= \frac{80(0)}{0^2 + 40}$

$= \frac{0}{40}$

$= 0$

The y-intercept is 0.

The end behavior is: as $x \to -\infty, f(x) \to 0$ and as $x \to \infty, f(x) \to 0$.

c. To find the week(s) during which 5 products sold, set $f(x) = 0$ and solve for x.

$f(x) = \frac{80x}{x^2 + 40}$

$5 = \frac{80x}{x^2 + 40}$

$5x^2 + 200 = 80x$

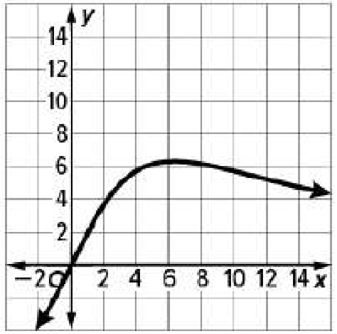
$5x^2 - 80x + 200 = 0$

$5(x^2 - 16x + 40) = 0$

Apply the quadratic formula to find $x \approx 3$ and 13.
So, 5 products were sold during weeks 3 and 13.

ANSWER:

a.



b. x-intercept: 0; y-intercept: 0; end behavior: As $x \to -\infty, f(x) \to 0$ and as $x \to \infty, f(x) \to 0$.

c. weeks 3 and 13

7. **FACTORY** The cost in cents to create a certain part of a small engine is modeled by $f(x) = \frac{18x - 12}{6x}$, where x is the number of parts made.

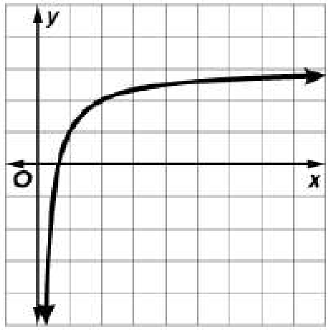
a. Graph the function.

b. Find the x-and y- intercepts and end behavior of the graph.

c. About how much does the 6th part cost to make?

SOLUTION:

a. Because the degree of $a(x)$ = the degree of $b(x)$, the horizontal asymptote is the line $y = \frac{\text{leading coefficient of } a(x)}{\text{leading coefficient of } b(x)}$, so $y = \frac{18}{6}$ or $y = 3$.



b. Find the x-intercept by setting $y = 0$.

$18x - 12 = 0$

$18x = 12$

$x = \frac{12}{18}$ or $\frac{2}{3}$

The x-intercept is $\frac{2}{3}$.

To find the y-intercept, setting $x = 0$ is undefined, so there is no y-intercept.
The end behavior is: as $x \to -\infty, f(x) \to 3$ and as $x \to \infty, f(x) \to 3$.

c. To find about how much it costs to make the 6th part, substitute 6 for x into the function equation.

$f(x) = \frac{18x - 12}{6x}$

$= \frac{18(6) - 12}{6(6)}$

$= \frac{108 - 12}{36}$

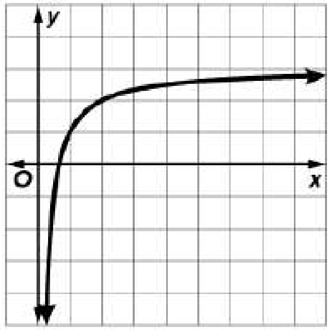
$= \frac{96}{36}$

≈ 2.7

It costs about 2.7 cents to make the 6th part.

ANSWER:

a.



b. x-intercept: $\frac{2}{3}$; y-intercept: none; end behavior:

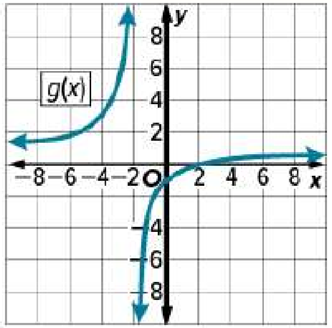
As $x \to -\infty, f(x) \to 3$ and as $x \to \infty, f(x) \to 3$.

c. $\approx$ 2.7 cents

For Exercises 8-10, consider the given function and the function shown in the graph.

a. Graph the given function.
b. Which function has the greater y-intercept?
c. Compare the asymptotes of the two functions.

8. $f(x) = \frac{x - 5}{3x + 5}$ and $g(x)$ shown in the graph.



SOLUTION:

a.

Step 1 Find the zeros.

5	Graph rational functions with vertical and horizontal asymptotes	1-10; Example 1 & 3	343; 337-338-339-340	Solution

Set $a(x) = 0$

$x - 5 = 0$

$x = 5$

There is a zero at $x = 5$.

Step 2 Find the asymptotes.

Find the vertical asymptote. Set $b(x) = 0$.

$3x + 5 = 0$

$3x = -5$

$x = -\frac{5}{3}$

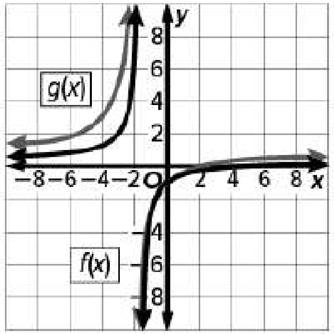
Because the degree of the numerator equals the degree of the denominator, the horizontal asymptote is the line

$y = \frac{\text{leading coefficient of } a(x)}{\text{leading coefficient of } b(x)}$, so $y = \frac{1}{3}$.

Step 3 Draw the graph.

Graph the asymptote. Then make a table of values, and graph.

x	$f(x)$
–5	1
–4	$\frac{9}{7}$
–3	2
–2	7
–1	0
0	–3
1	$-\frac{1}{2}$
2	$-\frac{3}{11}$
3	$-\frac{1}{7}$



b. They both have the same y-intercept at –1.

c. Vertical asymptotes: $f(x)$: $x = -\frac{5}{3}$

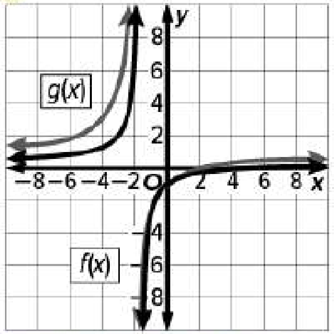
$g(x)$: $x = -2$

Horizontal asymptotes: $f(x)$: $y = \frac{1}{3}$ $g(x)$: $y =$

1

ANSWER:

a.



b. The intercepts are the same.

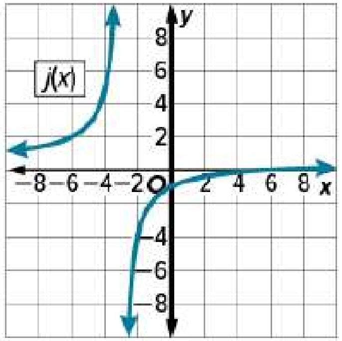
c. Vertical asymptotes: $f(x)$: $x = -\frac{5}{3}$

$g(x)$: $x = -2$

Horizontal asymptotes: $f(x)$: $y = \frac{1}{3}$ $g(x)$: $y =$

1

9. $h(x) = \frac{x+1}{4x-4}$ and $j(x)$ shown in the graph.



SOLUTION:

a.

Step 1 Find the zeros.

Set $a(x) = 0$

$x + 1 = 0$

$x = -1$

There is a zero at $x = -1$.

Step 2 Find the asymptotes.

Find the vertical asymptote. Set $b(x) = 0$.

$4x - 4 = 0$

$4x = 4$

$x = 1$

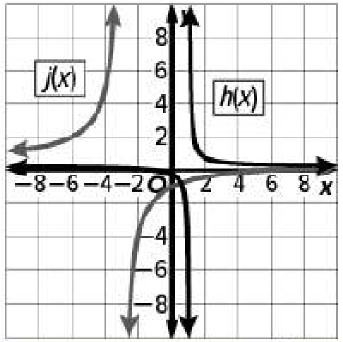
Because the degree of the numerator equals the degree of the denominator, the horizontal asymptote is the line

$y = \frac{\text{leading coefficient of } a(x)}{\text{leading coefficient of } b(x)}$, so $y = \frac{1}{4}$.

Step 3 Draw the graph.

Graph the asymptote. Then make a table of values, and graph.

x	$f(x)$
–4	$\frac{3}{20}$
–3	$\frac{1}{8}$
–2	$\frac{1}{2}$
–1	0
0	$-\frac{1}{4}$
2	$\frac{3}{4}$
3	$\frac{1}{2}$
4	$\frac{5}{16}$
5	$\frac{3}{8}$



b. $h(x)$ has a y-intercept at $-\frac{1}{4}$. $j(x)$ appears to have a y-intercept at $x = -1$, so $h(x)$ has a greater y-intercept.

c. Vertical asymptotes: $h(x)$: $x = 1$ $j(x)$: $x = -3$

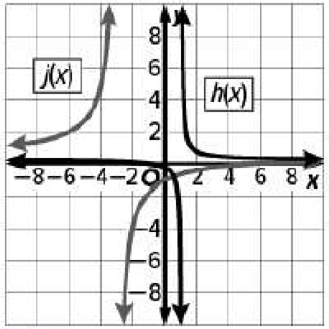
Horizontal aymptotes: $h(x)$: $y = \frac{1}{4}$ $j(x)$: $y = \frac{1}{2}$

ANSWER:

a.

5	Graph rational functions with vertical and horizontal asymptotes	1-10; Example 1 & 3	343; 337-338-339-340

Solution

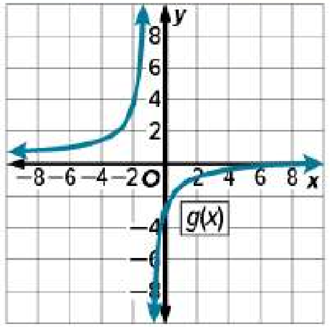


b. $h(x)$

c. Vertical asymptotes: $h(x): x = 1$ $j(x): x = -3$

Horizontal ayymptotes: $h(x): y = \frac{1}{4}$ $j(x): y = \frac{1}{2}$

10. $f(x) = \frac{x-3}{2x+7}$ and $g(x)$ shown in the graph.



SOLUTION:

a.

Step 1 Find the zeros.

Set $a(x) = 0$

$x - 3 = 0$

$x = 3$

There is a zero at $x = 3$.

Step 2 Find the asymptotes.

Find the vertical asymptote. Set $b(x) = 0$.

$2x + 7 = 0$

$2x = -7$

$x = -\frac{7}{2}$

Because the degree of the numerator equals the degree of the denominator, the horizontal asymptote is the line

$y = \frac{\text{leading coefficient of } a(x)}{\text{leading coefficient of } b(x)}$, so $y = \frac{1}{2}$.

Step 3 Draw the graph.

Graph the asymptote. Then make a table of values, and graph.

x	$f(x)$
−5	$2\frac{1}{3}$
−4	7
−3	−6
−2	$-1\frac{2}{3}$
−1	$-\frac{4}{5}$
0	$-\frac{3}{7}$
1	$-\frac{2}{9}$
2	$-\frac{1}{11}$
3	0

الاسئلة الالكترونية للامتحان

6	Find the composition of functions	21-35	165
7	Find the inverse of a function or relation	5-14	171
8	Write expressions with rational exponents in radical form and vice versa	1-12	179
9	Graph and analyze radical functions	27-39	191
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14	Evaluate expressions involving the natural base and natural logarithm	10-15	237
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17	Determine properties of reciprocal functions	51-56	336
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19	Recognize and solve direct and joint variation problems	19-22	351
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6	Find the composition of functions	21-35	165

If $f(x) = 3x$, $g(x) = x + 4$, and $h(x) = x^2 - 1$, find each value.

21. $f[g(1)]$

22. $g[h(0)]$

23. $g[f(-1)]$

26. $h[f(10)]$

6	Find the composition of functions	21-35	165

If $f(x) = 3x$, $g(x) = x + 4$, and $h(x) = x^2 - 1$, find each value.

27. $f[h(8)]$

28. $[f \circ (h \circ g)](1)$

29. $[f \circ (g \circ h)](-2)$

30. $h[f(-6)]$

6	Find the composition of functions	21-35	165

If $f(x) = 3x$, $g(x) = x + 4$, and $h(x) = x^2 - 1$, find each value.

31. $f[h(0)]$

33. $f[h(-2)]$

34. $[g \circ (f \circ h)](-1)$

35. $[h \circ (f \circ g)](3)$

6	Find the composition of functions	21-35	165

Solution

solution method

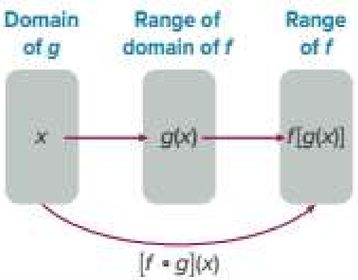
Lesson 4-1

Operations on Functions

Learn Compositions of Functions

Key Concept • Composition of Functions

Suppose f and g are functions such that the range of g is a subset of the domain of f. Then the composition function $f \circ g$ can be described by $[f \circ g](x) = f[g(x)]$.



Example 5 Compose Functions

If $f(x) = 3x$, $g(x) = x + 4$, and $h(x) = x^2 - 1$, find each value.

21. $f[g(1)]$

SOLUTION:

$f[g(1)] = f[1+4]$ Substitution.
$= f(5)$ Simplify.
$= 3(5)$ Substitution.
$= 15$ Simplify.

ANSWER:

15

22. $g[h(0)]$

SOLUTION:

$g[h(0)] = g[0^2 - 1]$ Substitution.
$= g(-1)$ Simplify.
$= -1 + 4$ Substitution.
$= 3$ Simplify.

ANSWER:

3

23. $g[f(-1)]$

SOLUTION:

$g[f(-1)] = g[3(-1)]$ Substitution.
$= g(-3)$ Simplify.
$= -3 + 4$ Substitution.
$= 1$ Simplify.

ANSWER:

1

24. $h[f(5)]$

SOLUTION:

$h[f(5)] = h[3(5)]$ Substitution.
$= h(15)$ Simplify.
$= 15^2 - 1$ Substitution.
$= 224$ Simplify.

ANSWER:

224

6	Find the composition of functions	21-35	165	Solution

25. $g[h(-3)]$

SOLUTION:

$g[h(-3)] = g[(-3)^2 - 1]$ Substitution.
$= g(8)$ Simplify.
$= 8 + 4$ Substitution.
$= 12$ Simplify.

ANSWER:
12

26. $h[f(10)]$

SOLUTION:

$h[f(10)] = h[3(10)]$ Substitution.
$= h(30)$ Simplify.
$= 30^2 - 1$ Substitution.
$= 899$ Simplify.

ANSWER:
899

27. $f[h(8)]$

SOLUTION:

$f[h(8)] = f[8^2 - 1]$ Substitution.
$= f(63)$ Simplify.
$= 3(63)$ Substitution.
$= 189$ Simplify.

ANSWER:
189

28. $[f \circ (h \circ g)](1)$

SOLUTION:

$[f \circ (h \circ g)](1) = f[h(g(1))]$ Compositi
$= f[h(1+4)]$ Substitutic
$= f[h(5)]$ Simplify.
$= f(5^2 - 1)$ Substitutic
$= f(24)$ Simplify.
$= 3(24)$ Substitutic
$= 72$ Simplify.

ANSWER:
72

29. $[f \circ (g \circ h)](-2)$

SOLUTION:

$[f \circ (g \circ h)](-2) = f[g(h(-2))]$ Co
$= f[g((-2)^2 - 1)]$ Sul
$= f[g(3)]$ Sin
$= f(3+4)$ Sul
$= f(7)$ Sin
$= 3(7)$ Sul
$= 21$ Sin

ANSWER:
21

30. $h[f(-6)]$

SOLUTION:

$h[f(-6)] = h[3(-6)]$ Substitution.
$= h(-18)$ Simplify.
$= (-18)^2 - 1$ Substitution.
$= 323$ Simplify.

ANSWER:
323

31. $f[h(0)]$

SOLUTION:

$f[h(0)] = f[0^2 - 1]$ Substitution.
$= f(-1)$ Simplify.
$= 3(-1)$ Substitution.
$= -3$ Simplify.

ANSWER:
–3

32. $f[g(7)]$

SOLUTION:

$f[g(7)] = f[7+4]$ Substitution.
$= f(11)$ Simplify.
$= 3(11)$ Substitution.
$= 33$ Simplify.

ANSWER:
33

33. $f[h(-2)]$

SOLUTION:

$f[h(-2)] = f[(-2)^2 - 1]$ Substitution.
$= f(3)$ Simplify.
$= 3(3)$ Substitution.
$= 9$ Simplify.

ANSWER:
9

34. $[g \circ (f \circ h)](-1)$

SOLUTION:

$[g \circ (f \circ h)](-1) = g[f(h(-1))]$ Composition
$= g[f((-1)^2 - 1)]$ Substitution.
$= g[f(0)]$ Simplify.
$= g(3(0))$ Substitution.
$= g(0)$ Simplify.
$= 0 + 4$ Substitution.
$= 4$ Simplify.

ANSWER:
4

35. $[h \circ (f \circ g)](3)$

SOLUTION:

$[h \circ (f \circ g)](3) = h[f(g(3))]$ Composition of func
$= h[f(3+4)]$ Substitution.
$= h[f(7)]$ Simplify.
$= h(3(7))$ Substitution.
$= h(21)$ Simplify.
$= 21^2 - 1$ Substitution.
$= 440$ Simplify.

ANSWER:
440

7	Find the inverse of a function or relation	5-14	171

Examples 2 and 3

Find the inverse of each function. Then graph the function and its inverse. If necessary, restrict the domain of the inverse so that it is a function.

5. $f(x) = x + 2$

6. $g(x) = 5x$

7	Find the inverse of a function or relation	5-14	171

7. $f(x) = -2x + 1$

8. $h(x) = \frac{x-4}{3}$

7	Find the inverse of a function or relation	5-14	171

9. $f(x) = -\frac{5}{3}x - 8$

10. $g(x) = x + 4$

7	Find the inverse of a function or relation	5-14	171

11. $f(x) = 4x$

12. $f(x) = -8x + 9$

7	Find the inverse of a function or relation	5-14	171

13. $f(x) = 5x^2$

14. $h(x) = x^2 + 4$

7	Find the inverse of a function or relation	5-14	171	Solution

solution method

Lesson 4-2

Inverse Relations and Functions

Learn Inverse Relations and Functions

Two relations are **inverse relations** if one relation contains elements of the form (a, b) when the other relation contains the elements of the form (b, a).

Two functions f and g are **inverse functions** if and only if both of their compositions are the identity function.

Key Concepts • Inverse Functions

Words: If f and f^{-1} are inverses, then $f(a) = b$ if and only if $f^{-1}(b) = a$.

Example: Let $f(x) = x - 5$ and represent its inverse as $f^{-1}(x) = x + 5$.

Evaluate $f(7)$.	Evaluate $f^{-1}(2)$.
$f(x) = x - 5$	$f^{-1}(x) = x + 5$
$f(7) = 7 - 5$ or 2	$f^{-1}(2) = 2 + 5$ or 7

Not all functions have an inverse function. If a function fails the horizontal line test, you can restrict the domain of the function to make the inverse a function. Choose a portion of the domain on which the function is one-to-one. There may be more than one possible domain.

Example 2 Inverse Functions

Example 3 Inverses with Restricted Domains

5. $f(x) = x + 2$

SOLUTION:

Rewrite the function as an equation relating x and y.

$f(x) = x + 2 \rightarrow y = x + 2$

Exchange x and y.

$x = y + 2$

Solve for y.

$x = y + 2$

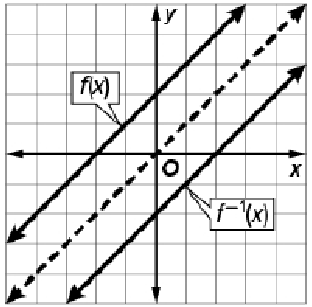
$x - 2 = y$

Replace y with $f^{-1}(x)$ in the equation.

$y = x - 2 \rightarrow f^{-1}(x) = x - 2$

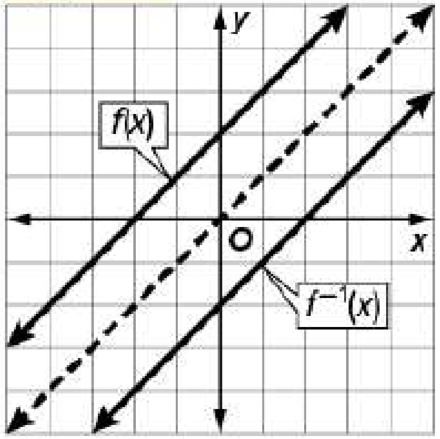
The inverse of $f(x) = x + 2$ is $f^{-1}(x) = x - 2$.

Graph $f(x)$ and $f^{-1}(x)$.



ANSWER:

$f^{-1}(x) = x - 2$



7	Find the inverse of a function or relation	5-14	171	Solution

6. $g(x) = 5x$

SOLUTION:

Rewrite the function as an equation relating x and y.

$g(x) = 5x \rightarrow y = 5x$

Exchange x and y.

$x = 5y$

Solve for y.

$x = 5y$

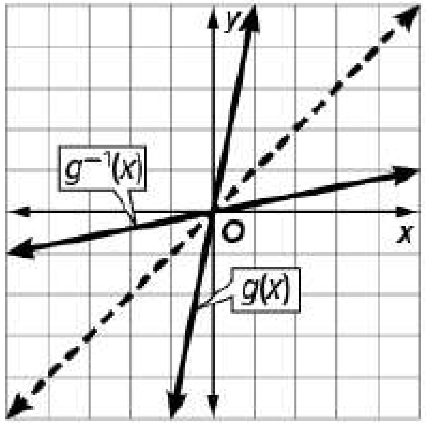
$\frac{1}{5}x = y$

Replace y with $g^{-1}(x)$ in the equation.

$y = \frac{1}{5}x \rightarrow g^{-1}(x) = \frac{1}{5}x$

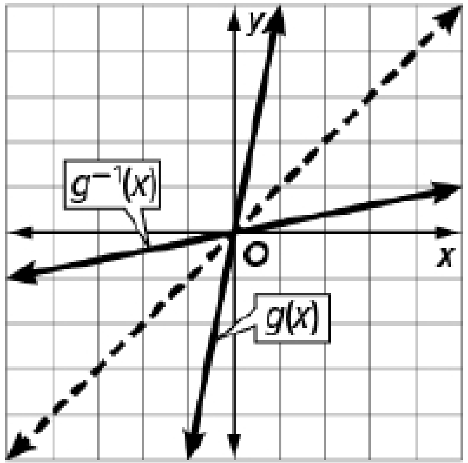
The inverse of $g(x) = 5x$ is $g^{-1}(x) = \frac{1}{5}x$.

Graph $g(x)$ and $g^{-1}(x)$.



ANSWER:

$g^{-1}(x) = \frac{1}{5}x$



7. $f(x) = -2x + 1$

SOLUTION:

Rewrite the function as an equation relating x and y.

$f(x) = -2x + 1 \rightarrow y = -2x + 1$

Exchange x and y.

$x = -2y + 1$

Solve for y.

$x = -2y + 1$

$x - 1 = -2y$

$\frac{x-1}{-2} = y$

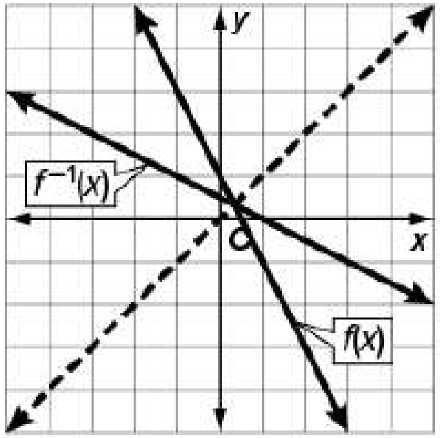
$-\frac{x}{2} + \frac{1}{2} = y$

Replace y with $f^{-1}(x)$ in the equation.

$y = -\frac{x}{2} + \frac{1}{2} \rightarrow f^{-1}(x) = -\frac{x}{2} + \frac{1}{2}$

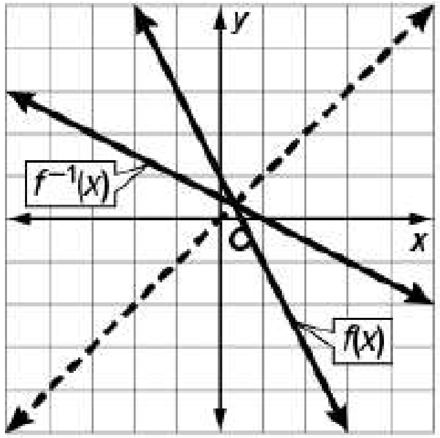
The inverse of $f(x) = -2x + 1$ is $f^{-1}(x) = -\frac{x}{2} + \frac{1}{2}$.

Graph $f(x)$ and $f^{-1}(x)$.



ANSWER:

$f^{-1}(x) = -\frac{x}{2} + \frac{1}{2}$



7	Find the inverse of a function or relation	5-14	171	Solution

8. $h(x) = \frac{x-4}{3}$

SOLUTION:

Rewrite the function as an equation relating x and y.

$h(x) = \frac{x-4}{3} \to y = \frac{x-4}{3}$

Exchange x and y.

$x = \frac{y-4}{3}$

Solve for y.

$x = \frac{y-4}{3}$

$3x = y - 4$

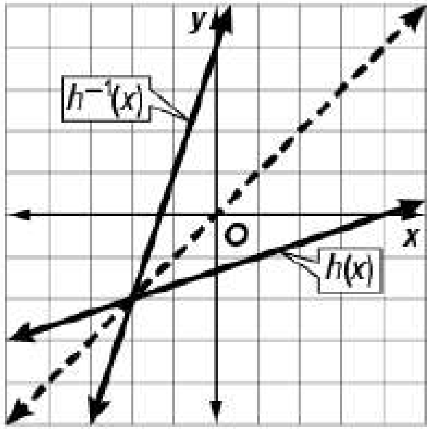
$3x + 4 = y$

Replace y with $h^{-1}(x)$ in the equation.

$y = 3x + 4 \to h^{-1}(x) = 3x + 4$

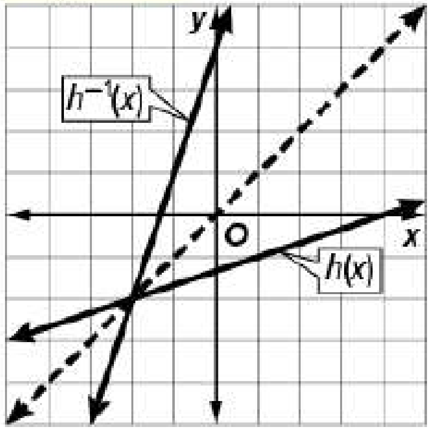
The inverse of $h(x) = \frac{x-4}{3}$ is $h^{-1}(x) = 3x + 4$.

Graph $h(x)$ and $h^{-1}(x)$.



ANSWER:

$h^{-1}(x) = 3x + 4$



9. $f(x) = -\frac{5}{3}x - 8$

SOLUTION:

Rewrite the function as an equation relating x and y.

$f(x) = -\frac{5}{3}x - 8 \to y = -\frac{5}{3}x - 8$

Exchange x and y.

$x = -\frac{5}{3}y - 8$

Solve for y.

$x = -\frac{5}{3}y - 8$

$x + 8 = -\frac{5}{3}y$

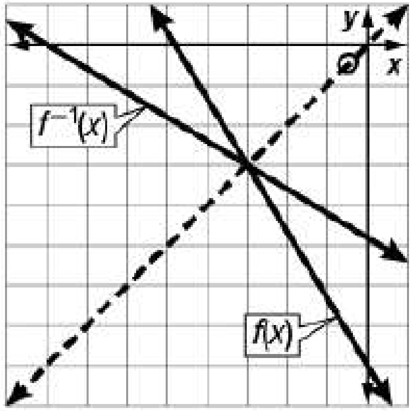
$-\frac{3}{5}(x + 8) = y$

Replace y with $f^{-1}(x)$ in the equation.

$y = -\frac{3}{5}(x + 8) \to f^{-1}(x) = -\frac{3}{5}(x + 8)$

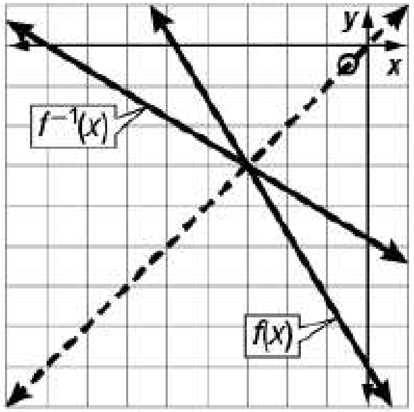
The inverse of $f(x) = -\frac{5}{3}x - 8$ is $f^{-1}(x) = -\frac{3}{5}(x + 8)$.

Graph $f(x)$ and $f^{-1}(x)$.



ANSWER:

$f^{-1}(x) = -\frac{3}{5}(x + 8)$



7	Find the inverse of a function or relation	5-14	171	Solution

10. $g(x) = x + 4$

SOLUTION:

Rewrite the function as an equation relating x and y.

$g(x) = x + 4 \rightarrow y = x + 4$

Exchange x and y.

$x = y + 4$

Solve for y.

$x = y + 4$

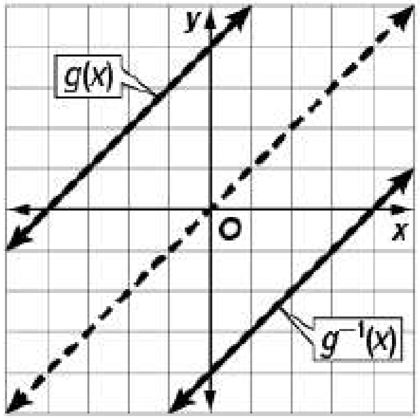
$x - 4 = y$

Replace y with $g^{-1}(x)$ in the equation.

$y = x - 4 \rightarrow g^{-1}(x) = x - 4$

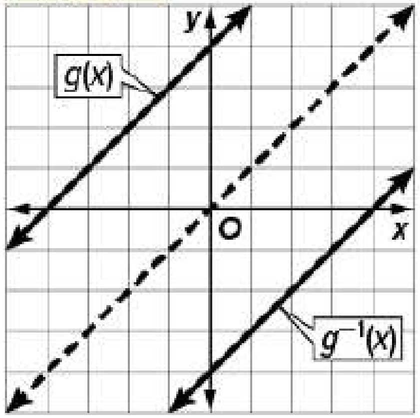
The inverse of $g(x) = x + 4$ is $g^{-1}(x) = x - 4$.

Graph $g(x)$ and $g^{-1}(x)$.



ANSWER:

$g^{-1}(x) = x - 4$



11. $f(x) = 4x$

SOLUTION:

Rewrite the function as an equation relating x and y.

$f(x) = 4x \rightarrow y = 4x$

Exchange x and y.

$x = 4y$

Solve for y.

$x = 4y$

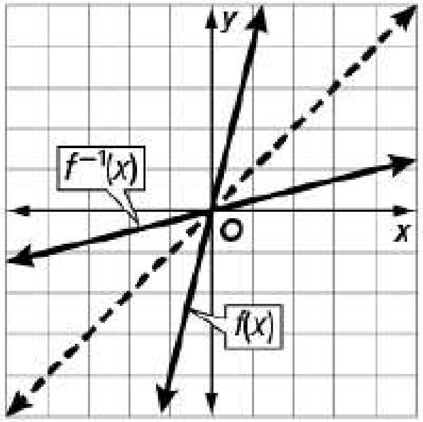
$\frac{1}{4}x = y$

Replace y with $f^{-1}(x)$ in the equation.

$y = \frac{1}{4}x \rightarrow f^{-1}(x) = \frac{1}{4}x$

The inverse of $f(x) = 4x$ is $f^{-1}(x) = \frac{1}{4}x$.

Graph $f(x)$ and $f^{-1}(x)$.



ANSWER:

$f^{-1}(x) = \frac{1}{4}x$

7	Find the inverse of a function or relation	5-14	171

Solution

12. $f(x) = -8x + 9$

SOLUTION:

Rewrite the function as an equation relating x and y.

$f(x) = -8x + 9 \rightarrow y = -8x + 9$

Exchange x and y.

$x = -8y + 9$

Solve for y.

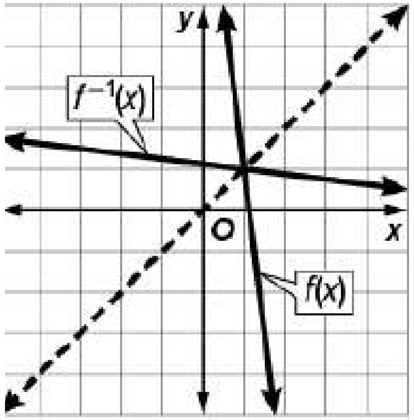
$$x = -8y + 9$$
$$x - 9 = -8y$$
$$\frac{x-9}{-8} = y$$
$$-\frac{x}{8} + \frac{9}{8} = y$$

Replace y with $f^{-1}(x)$ in the equation.

$$y = -\frac{x}{8} + \frac{9}{9} \rightarrow f^{-1}(x) = -\frac{x}{8} + \frac{9}{8}$$

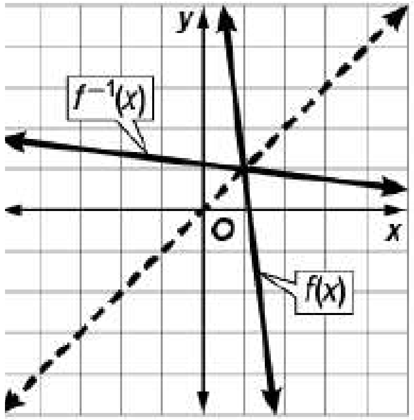
The inverse of $f(x) = -8x + 9$ is $f^{-1}(x) = -\frac{x}{8} + \frac{9}{8}$.

Graph $f(x)$ and $f^{-1}(x)$.



ANSWER:

$f^{-1}(x) = -\frac{x}{8} + \frac{9}{8}$



13. $f(x) = 5x^2$

SOLUTION:

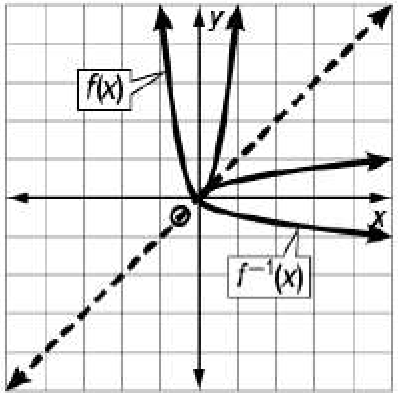
Find the inverse of $f(x)$.

$f(x) = 5x^2$	Original function
$y = 5x^2$	Replace $f(x)$ with y.
$x = 5y^2$	Exchange x and y.
$\frac{1}{5}x = y^2$	Divide each side by 5.
$\pm\sqrt{\frac{1}{5}x} = y$	Take the square root of each side.
$f^{-1}(x) = \pm\sqrt{\frac{1}{5}x}$	Replace y with $f^{-1}(x)$.
$f^{-1}(x) = \pm\frac{\sqrt{5x}}{5}$	Simplify.

Because $f(x)$ fails the horizontal line test, $f^{-1}(x)$ is not a function. Find the restricted domain of $f(x)$ so that $f^{-1}(x)$ will be a function. Look for a portion of the graph that is one-to-one. If the domain of $f(x)$ is restricted to $(-\infty, 0]$, then the inverse is $f^{-1}(x) = -\frac{\sqrt{5x}}{5}$. If the domain of $f(x)$ is restricted to $[0, \infty)$, then the inverse is $f^{-1}(x) = \frac{\sqrt{5x}}{5}$.

7	Find the inverse of a function or relation	5-14	171

Solution



ANSWER:

If the domain of $f(x)$ is restricted to $(-\infty, 0]$, then the inverse is $f^{-1}(x) = -\frac{\sqrt{5x}}{5}$.

If the domain of $f(x)$ is restricted to $[0, \infty)$, then the inverse is $f^{-1}(x) = \frac{\sqrt{5x}}{5}$.

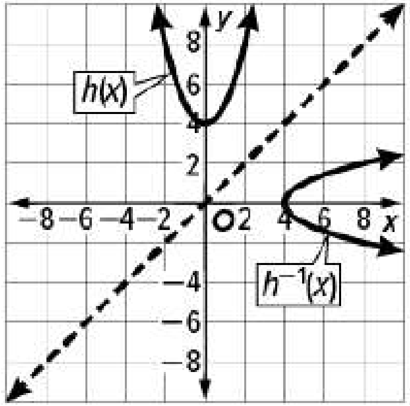
14. $h(x) = x^2 + 4$

SOLUTION:

Find the inverse of $h(x)$.

$h(x) = x^2 + 4$	Original function
$y = x^2 + 4$	Replace $h(x)$ with y.
$x = y^2 + 4$	Exchange x and y.
$x - 4 = y^2$	Subtract 4 from each side.
$\pm\sqrt{x-4} = y$	Take the square root of each side.
$h^{-1}(x) = \pm\sqrt{x-4}$	Replace y with $h^{-1}(x)$.

Because $h(x)$ fails the horizontal line test, $h^{-1}(x)$ is not a function. Find the restricted domain of $h(x)$ so that $h^{-1}(x)$ will be a function. Look for a portion of the graph that is one-to-one. If the domain of $h(x)$ is restricted to $(-\infty, 0]$, then the inverse is $h^{-1}(x) = -\sqrt{x-4}$. If the domain of $h(x)$ is restricted to $[0, \infty)$, then the inverse is $h^{-1}(x) = \sqrt{x-4}$.



ANSWER:

If the domain of $h(x)$ is restricted to $(-\infty, 0]$, $h^{-1}(x) = -\sqrt{x-4}$.

If the domain of $h(x)$ is restricted to $[0, \infty)$, $h^{-1}(x) = \sqrt{x-4}$.

8	Write expressions with rational exponents in radical form and vice versa	1-12	179

Examples 1 and 2

Simplify.

1. $\pm\sqrt{121x^4y^{16}}$

2. $\pm\sqrt{225a^{16}b^{36}}$

3. $\pm\sqrt{49x^4}$

4. $-\sqrt{16c^4d^2}$

5. $-\sqrt{81a^{16}b^{20}c^{12}}$

6. $-\sqrt{400x^{32}y^{40}}$

7. $\sqrt[4]{16(x-3)^{12}}$

8. $\sqrt[8]{x^{16}y^8}$

8	Write expressions with rational exponents in radical form and vice versa	1-12	179

Examples 1 and 2

Simplify.

9. $\sqrt[4]{81(x-4)^4}$

10. $\sqrt[6]{x^{18}}$

11. $\sqrt[4]{a^{12}}$

12. $\sqrt[3]{a^{12}}$

8	Write expressions with rational exponents in radical form and vice versa	1-12	179

Solution

solution method

Lesson 4-3

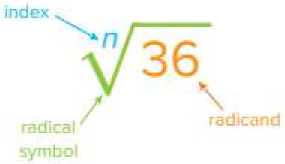
nth Roots and Rational Exponents

Learn *n*th Roots

Finding the square root of a number and squaring a number are inverse operations. To find the square root of a, you must find a number with a square of a. The inverse of raising a number to the nth power is finding the **nth root** of a number. The symbol $\sqrt[n]{\ }$ indicates an nth root.

For any real numbers a and b and any positive integer n, if $a^n = b$, then a is an nth root of b. For example, because $(-2)^6 = 64$, -2 is a sixth root of 64 and 2 is a principal root.

An example of an nth root is $\sqrt[n]{36}$, which is read as *the nth root of 36*. In this example, n is the **index** and 36 is the **radicand**, or the expression under the radical symbol.



Some numbers have more than one real nth root. For example, 16 has two square roots, 4 and -4, because 4^2 and $(-4)^2$ both equal 16. When there is more than one real root and n is even, the nonnegative root is called the **principal root**.

Key Concept • Real *n*th Roots

Suppose n is an integer greater than 1, a is a real number, and a is an nth root of b.

a	n is even.	n is odd.
$a > 0$	1 unique positive and 1 unique negative real root: $\pm\sqrt[n]{a}$	1 unique positive and 0 negative real root: $\sqrt[n]{a}$
$a < 0$	0 real roots	0 positive and 1 negative real root: $\sqrt[n]{a}$
$a = 0$	1 real root: $\sqrt[n]{0} = 0$	1 real root: $\sqrt[n]{0} = 0$

A radical expression is simplified when the radicand contains no fractions and no radicals appear in the denominator.

Example 1 Find Roots

Example 2 Simplify Using Absolute Value

Simplify.

1. $\pm\sqrt{121x^4y^{16}}$

SOLUTION:

$$\pm\sqrt{121x^4y^{16}} = \pm\sqrt{(11x^2y^8)^2} = \pm 11x^2y^8$$

ANSWER:

$\pm 11x^2y^8$

2. $\pm\sqrt{125a^{16}b^{36}}$

SOLUTION:

$$\pm\sqrt{125a^{16}b^{36}} = \pm\sqrt{(15a^8b^{18})^2} = \pm 15a^8b^{18}$$

ANSWER:

$\pm 15a^8b^{18}$

3. $\pm\sqrt{49x^4}$

SOLUTION:

$$\pm\sqrt{49x^4} = \pm\sqrt{(7x^2)^2} = \pm 7x^2$$

ANSWER:

$\pm 7x^2$

4. $-\sqrt{16c^4d^2}$

SOLUTION:

$$-\sqrt{16c^4d^2} = -\sqrt{(4c^2d)^2} = -4c^2|d|$$

Since d could be negative, you must use the absolute value of d to ensure that the principal square root is nonnegative.

ANSWER:

$-4c^2|d|$

5. $-\sqrt{81a^{16}b^{20}c^{12}}$

SOLUTION:

$$-\sqrt{81a^{16}b^{20}c^{12}} = -\sqrt{(9a^8b^{10}c^6)^2} = -9a^8b^{10}c^6$$

ANSWER:

$-9a^8b^{10}c^6$

6. $-\sqrt{400x^{32}y^{40}}$

SOLUTION:

$$-\sqrt{400x^{32}y^{40}} = -\sqrt{(20x^{16}y^{20})^2} = -20x^{16}y^{20}$$

ANSWER:

$-20x^{16}y^{20}$

7. $\sqrt[4]{16(x-3)^{12}}$

SOLUTION:

$$\sqrt[4]{16(x-3)^{12}} = \sqrt[4]{16} \cdot \sqrt[4]{(x-3)^{12}} = 2|(x-3)^3|$$

Since $(x-3)^3$ could be negative, you must use the absolute value of $(x-3)^3$ to ensure that the principal square root is nonnegative.

ANSWER:

$2|(x-3)^3|$

8. $\sqrt[8]{x^{16}y^8}$

SOLUTION:

$$\sqrt[8]{x^{16}y^8} = \sqrt[8]{(x^2y)^8} = x^2|y|$$

Because y could be negative, you must use the absolute value of y to ensure that the principal root is nonnegative.

ANSWER:

$x^2|y|$

8	Write expressions with rational exponents in radical form and vice versa	1-12	179

Solution

9. $\sqrt[4]{81(x-4)^4}$

SOLUTION:

$$\sqrt[4]{81(x-4)^4} = \sqrt[4]{81} \cdot \sqrt[4]{(x-4)^4}$$
$$= 3|x-4|$$

Since $(x-4)$ could be negative, you must use the absolute value of $(x-4)$ to ensure that the principal square root is nonnegative.

ANSWER:

$3|x-4|$

10. $\sqrt[6]{x^{18}}$

SOLUTION:

$$\sqrt[6]{x^{18}} = \sqrt[6]{(x^3)^6}$$
$$= |x^3|$$

Because x^3 could be negative, you must use the absolute value of x^3 to ensure that the principal root is nonnegative.

ANSWER:

$|x^3|$

11. $\sqrt[4]{a^{12}}$

SOLUTION:

$$\sqrt[4]{a^{12}} = \sqrt[4]{(a^3)^4}$$
$$= |a^3|$$

Because a^3 could be negative, you must use the absolute value of a^3 to ensure that the principal root is nonnegative.

ANSWER:

$|a^3|$

12. $\sqrt[3]{a^{12}}$

SOLUTION:

$$\sqrt[3]{a^{12}} = \sqrt[3]{(a^4)^3}$$
$$= a^4$$

ANSWER:

a^4

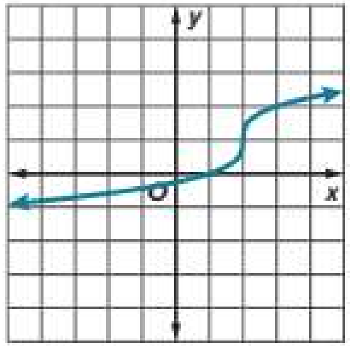
Write each expression in radical form, or write each radical in exponential form.

9	Graph and analyze radical functions	27-39	191

Example 7

Write a radical function for each graph.

27.



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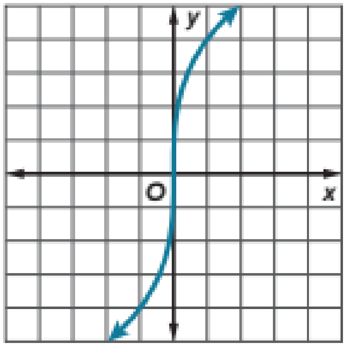
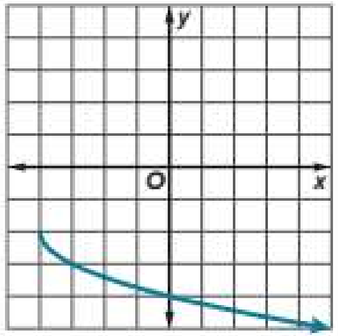
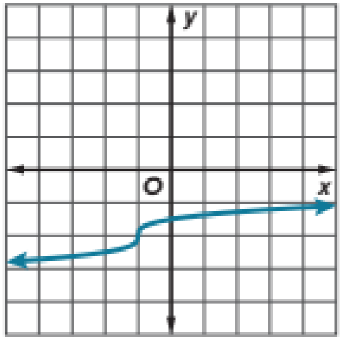


9	Graph and analyze radical functions	27-39	191

Write a radical function for each graph.

37. 

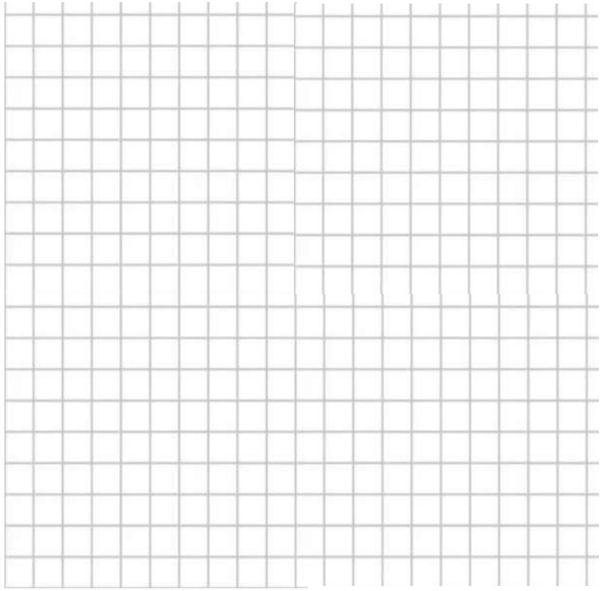
38. 

9	Graph and analyze radical functions	27-39	191

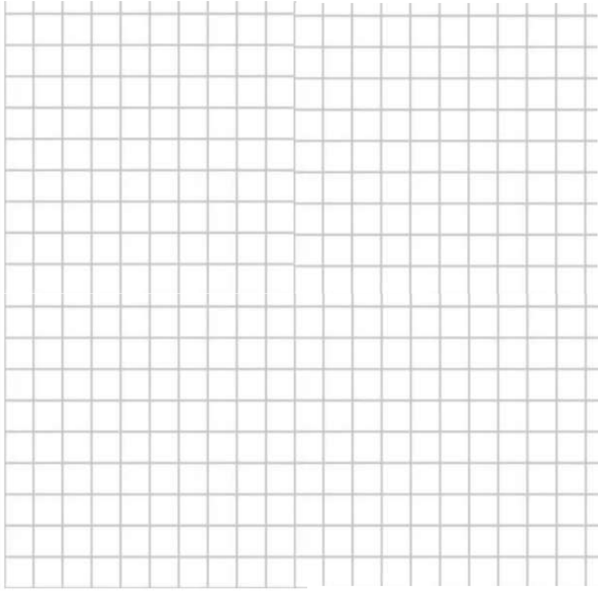
Mixed Exercises

Graph each function and state the domain and range. Then describe how it is related to the graph of the parent function.

29. $f(x) = 2\sqrt{x-5} - 6$



30. $f(x) = \frac{3}{4}\sqrt{x+12} + 3$

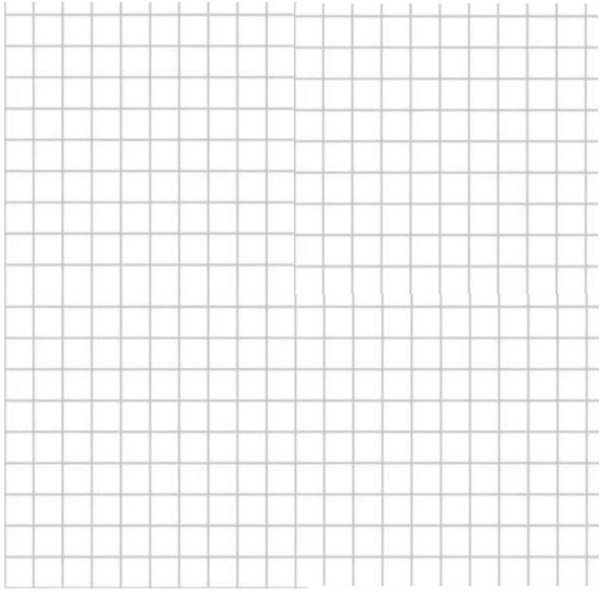


9	Graph and analyze radical functions	27-39	191

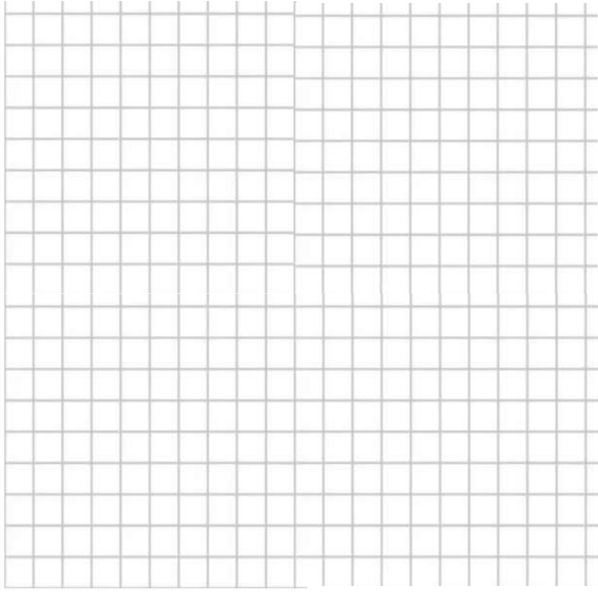
Mixed Exercises

Graph each function and state the domain and range. Then describe how it is related to the graph of the parent function.

31. $f(x) = -\frac{1}{5}\sqrt{x-1} - 4$



32. $f(x) = -3\sqrt{x+7} + 9$



9	Graph and analyze radical functions	27-39	191

Mixed Exercises

Graph each function and state the domain and range. Then describe how it is related to the graph of the parent function.

33. $f(x) = -\frac{1}{3}\sqrt[3]{x+2} - 3$

34. $f(x) = -\frac{1}{2}\sqrt[3]{2x-1} + 3$

9	Graph and analyze radical functions	27-39	191

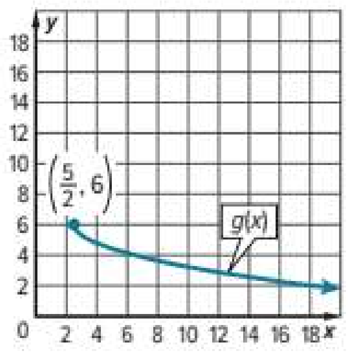
Graph each inequality.

35. $y \leq 6 - 3\sqrt{x - 4}$

36. $y < \sqrt{4x - 12} + 8$

9	Graph and analyze radical functions	27-39	191

39. **STRUCTURE** Consider the function $f(x) = -\sqrt{x+3} + \frac{13}{2}$ and the function $g(x)$ shown in the graph.



a. Determine which function has the greater maximum value. Explain your reasoning.

b. Compare the domains of the two functions.

c. Compare the average rates of change of the two functions over the interval [6, 13].

9	Graph and analyze radical functions	27-39	191

solution method

Lesson 4-4

Graphing Radical Functions

Example 2 Graph a Transformed Square Root Function

Example 6 Compare Radical Functions

Example 7 Write a Radical Function

Learn Graphing Square Root Functions

A **radical function** is a function that contains radicals with variables in the radicand. One type of radical function is a **square root function**, which is a function that contains the square root of a variable expression.

Key Concept • Parent Function of Square Root Functions

The parent function of the square root functions is $f(x) = \sqrt{x}$.

Domain: $\{x \mid x \geq 0\}$

Range: $\{f(x) \mid f(x) \geq 0\}$

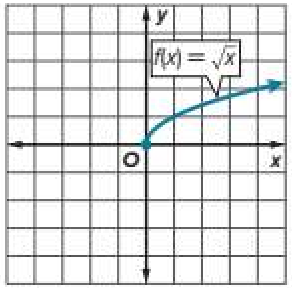
Intercepts: $x = 0, f(x) = 0$

End behavior: As $x \to 0, f(x) \to 0$, and as $x \to \infty, f(x) \to \infty$.

Increasing/decreasing: increasing when $x > 0$

Positive/negative: positive for $x > 0$

Symmetry: no symmetry



A square root function can be written in the form $g(x) = a\sqrt{x - h} + k$. Each constant in the equation affects the parent graph.

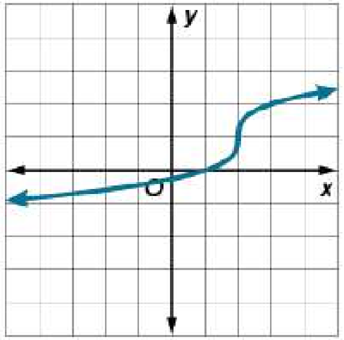
- The value of $|a|$ stretches or compresses (dilates) the parent graph.
- When the value of a is negative, the graph is reflected in the x-axis.
- The value of h shifts (translates) the parent graph left or right.
- The value of k shifts (translates) the parent graph up or down.

9	Graph and analyze radical functions	27-39	191

Solution

Write a radical function for each graph.

27.



SOLUTION:

Identify the index.

Because the domain and range is all real numbers, the index is odd. This function can be represented by $f(x) = a\sqrt[3]{x-h} + k$.

Identify any transformations.

The function has been translated 2 units right and 1 unit up.

Therefore, $h = 2$ and $k = 1$. To find the value of a, use a point as well as the values of h and k.

$f(x) = a\sqrt[3]{x-h} + k$	Cube root function
$1 = a\sqrt[3]{2-2} + 1$	$h = 2$, $k = 1$, $(x, f(x)) = (2, 1)$
$1 = a \cdot 1$	Simplify.
$1 = a$	Simplify.

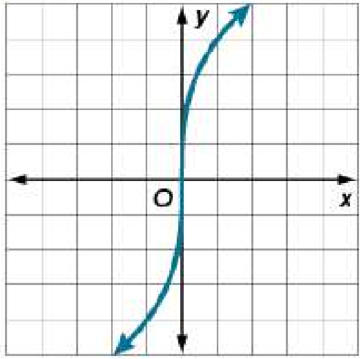
Write the function.

Substitute the values of a, h, and k to write the function. The graph is represented by $f(x) = \sqrt[3]{x-2} + 1$.

ANSWER:

$f(x) = \sqrt[3]{x-2} + 1$

28.



SOLUTION:

Identify the index.

Because the domain and range is all real numbers, the index is odd. This function can be represented by $f(x) = a\sqrt[3]{x-h} + k$.

Identify any transformations.

The function has not been translated.

Therefore, $h = 0$ and $k = 0$. To find the value of a, use a point as well as the values of h and k.

$f(x) = a\sqrt[3]{x-h} + k$	Cube root function
$4 = a\sqrt[3]{1-0} + 0$	$h = 0$, $k = 0$, $(x, f(x)) = (1, 4)$
$4 = a \cdot 1$	Simplify.
$4 = a$	Simplify.

Write the function.

Substitute the values of a, h, and k to write the function. The graph is represented by $f(x) = 4\sqrt[3]{x}$.

ANSWER:

$f(x) = 4\sqrt[3]{x}$

Graph each function and state the domain and range. Then describe how it is related to the graph of the parent function.

29. $f(x) = 2\sqrt{x-5} - 6$

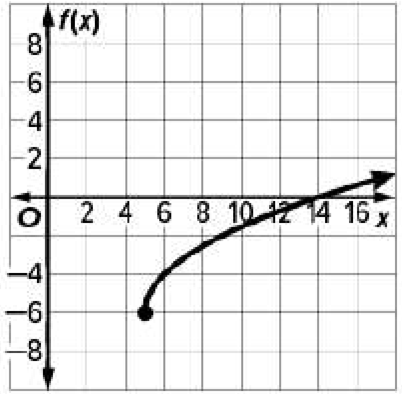
SOLUTION:

Step 1: Determine the minimum domain value.

$x - 5 \geq 0$	Write an inequality using the radicand.
$x \geq 5$	Simplify.

Step 2: Make a table. Use the x-values determined from **Step 1** to make a table.

x	$f(x)$
5	–6
6	–4
7	≈ –3.17
8	≈ –2.54
9	–2
10	≈ –1.53
11	≈ –1.10
12	≈ –0.71
13	≈ –0.34
14	0
15	≈ 0.32



The domain is $\{x \mid x \geq 5\}$ and the range is $\{f(x) \mid f(x) \geq -6\}$.

Step 3: Compare to the parent function.

The minimum is (5, –6).

Because the parent function is $y = \sqrt{x}$, the transformed function is $f(x) = a\sqrt{x-h} + k$ where

$a = 2$, $h = 5$, and $k = -6$.

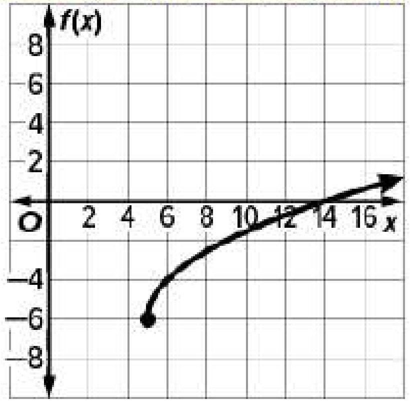
$a > 0$ and $|a| > 1$, so the graph of $y = \sqrt{x}$ is stretched vertically by a factor of $|a|$, or 2.

$h > 0$, so the graph is then translated right h units, or 5 units.

$k < 0$, so the graph is then translated down k units, or 6 units.

ANSWER:

D = $\{x \mid x \geq 5\}$; R = $\{f(x) \mid f(x) \geq -6\}$; stretched vertically, translated 5 units right and 6 units down;



9	Graph and analyze radical functions	27-39	191

Solution

30. $f(x) = \frac{3}{4}\sqrt{x+12} + 3$

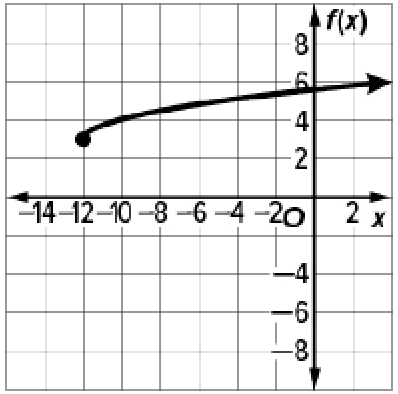
SOLUTION:

Step 1: Determine the minimum domain value.

$x + 12 \geq 0$ Write an inequality using the radicand.
$x \geq -12$ Simplify.

Step 2: Make a table. Use the x-values determined from **Step 1** to make a table.

x	$f(x)$
–12	3
–10	≈ 4.06
–8	4.5
–6	≈ 4.84
–4	≈ 5.12
–2	≈ 5.37
0	≈ 5.60
2	≈ 5.81
4	6



The domain is $\{x \mid x \geq -12\}$ and the range is $\{f(x) \mid f(x) \geq 3\}$.

Step 3: Compare to the parent function.

The minimum is (–12, 3).

Because the parent function is $y = \sqrt{x}$, the transformed function is $f(x) = a\sqrt{x-h} + k$ where $a = \frac{3}{4}$, $h = -12$, and $k = 3$.

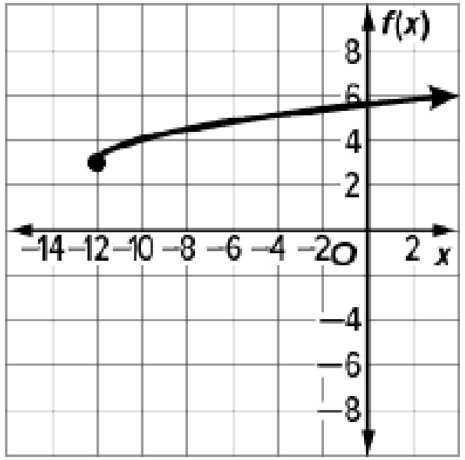
$a > 0$ and $0 < |a| < 1$, so the graph of $y = \sqrt{x}$ is compressed vertically by a factor of $|a|$, or $\frac{3}{4}$.

$h < 0$, so the graph is then translated left h units, or 12 units.

$k > 0$, so the graph is then translated up k units, or 3 units.

ANSWER:

D = $\{x \mid x \geq -12\}$; R = $\{f(x) \mid f(x) \geq 3\}$; compressed vertically, translated 12 units left and 3 units up;



31. $f(x) = -\frac{1}{5}\sqrt{x-1} - 4$

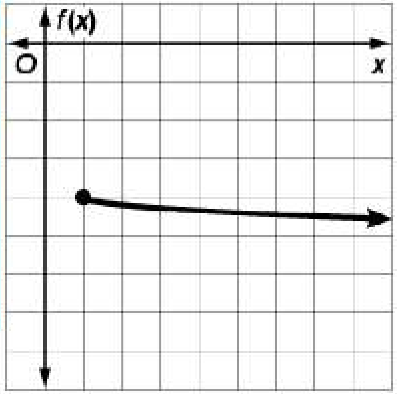
SOLUTION:

Step 1: Determine the minimum domain value.

$x - 1 \geq 0$ Write an inequality using the radicand.
$x \geq 1$ Simplify.

Step 2: Make a table. Use the x-values determined from **Step 1** to make a table.

x	$f(x)$
1	–4
2	–4.2
3	≈ – 4.28
4	≈ – 4.35
5	–4.4
6	≈ – 4.45



The domain is $\{x \mid x \geq 1\}$ and the range is $\{f(x) \mid f(x) \leq -4\}$.

Step 3: Compare to the parent function.

The maximum is (1, –4).

Because the parent function is $y = \sqrt{x}$, the transformed function is $f(x) = a\sqrt{x-h} + k$ where $a = -\frac{1}{5}$, $h = 1$, and $k = -4$.

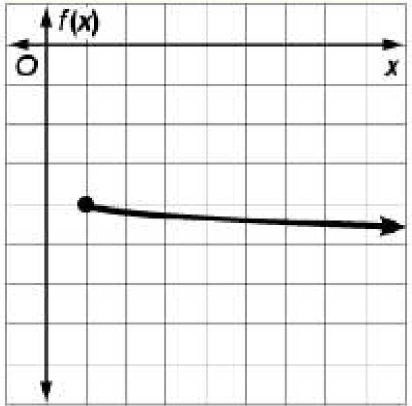
$a < 0$ and $0 < |a| < 1$, so the graph of $y = \sqrt{x}$ is reflected across the x-axis and compressed vertically by a factor of $|a|$, or $\frac{1}{5}$.

$h > 0$, so the graph is then translated right $|h|$ units, or 1 unit.

$k < 0$, so the graph is then translated down k units, or 4 units.

ANSWER:

D = $\{x \mid x \geq 1\}$; R = $\{f(x) \mid f(x) \leq -4\}$; compressed vertically, translated 1 unit right and 4 units down, reflected in the x-axis;



9	Graph and analyze radical functions	27-39	191	**Solution**

32. $f(x) = -3\sqrt{x+7}+9$

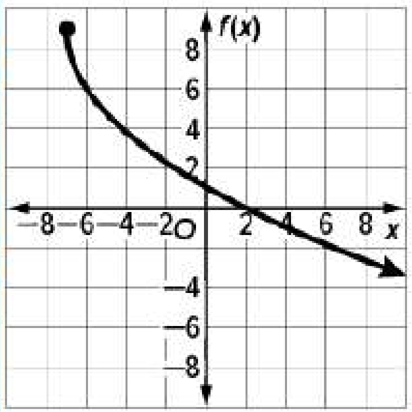
SOLUTION:

Step 1: Determine the minimum domain value.

$x+7 \geq 0$ Write an inequality using the radicand.
$x \geq -7$ Simplify.

Step 2: Make a table. Use the x-values determined from **Step 1** to make a table.

x	$f(x)$
–7	9
–6	6
–5	≈ 4.76
–4	≈ 3.80
–3	3
–2	≈ 2.29
–1	≈ 2.65
0	≈ 1.06
1	≈ 0.51
2	0
3	≈ –0.49
4	≈ –0.95
5	≈ –1.39
6	≈ –1.82



The domain is $\{x \mid x \geq -7\}$ and the range is $\{f(x) \mid f(x) \leq 9\}$.

The maximum is (–7, 9).

Because the parent function is $y=\sqrt{x}$, the transformed function is $f(x)=a\sqrt{x-h}+k$ where $a=-3$, $h=-7$, and $k=9$.

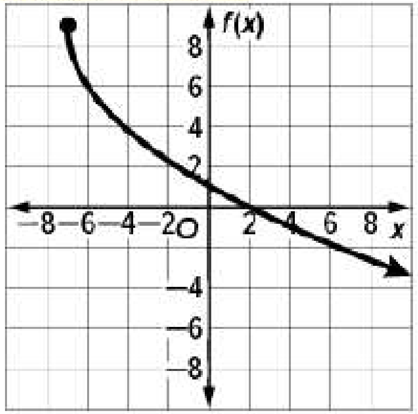
$a<0$ and $|a|>1$, so the graph of $y=\sqrt{x}$ is reflected across the x-axis and stretched vertically by a factor of $|a|$, or 3.

$h<0$, so the graph is then translated left $|h|$ units, or 7 units.

$k>0$, so the graph is then translated up k units, or 9 units.

ANSWER:

D = $\{x \mid x \geq -7\}$; R = $\{f(x) \mid f(x) \leq 9\}$; stretched vertically, translated 7 units left and 9 units up, reflected in the x-axis;

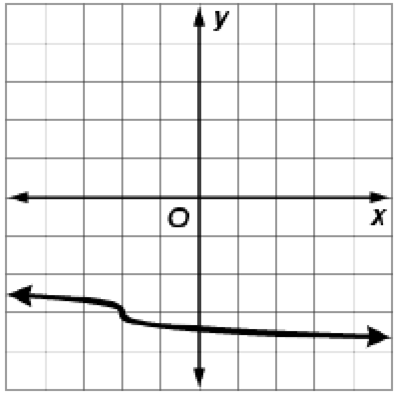


33. $f(x) = -\frac{1}{3}\sqrt[3]{x+2}-3$

SOLUTION:

Make a table and graph the function.

x	$f(x)$
–4	–2.58
–3	≈ –2.67
–2	–3
–1	≈ –3.33
0	–3.42



The domain is all real numbers, or (–∞, ∞), and the range is all real numbers, or (–∞, ∞).

Compare to the parent function.

Because the parent function is $y=\sqrt[3]{x}$, the transformed function is $f(x)=a\sqrt[3]{x-h}+k$ where $a=-\frac{1}{3}$, $h=-2$, and $k=-3$.

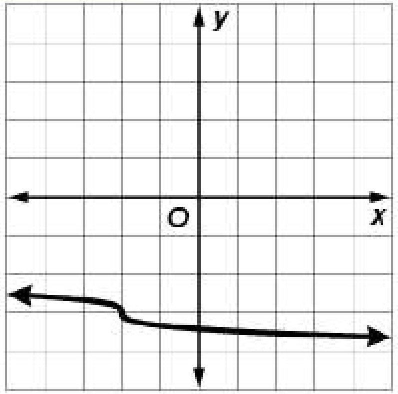
$a<0$ and $0<|a|<1$, so the graph of $y=\sqrt[3]{x}$ is reflected across the x-axis and compressed vertically by a factor of $|a|$, or $\frac{1}{3}$.

$h<0$, so the graph is then translated left $|h|$ units, o 2 units.

$k<0$, so the graph is then translated down k units, or 3 units.

ANSWER:

D = (–∞, ∞); R = (–∞, ∞); compressed vertically, translated 2 units left and 3 units down, reflected in the x-axis;



9	Graph and analyze radical functions	27-39	191

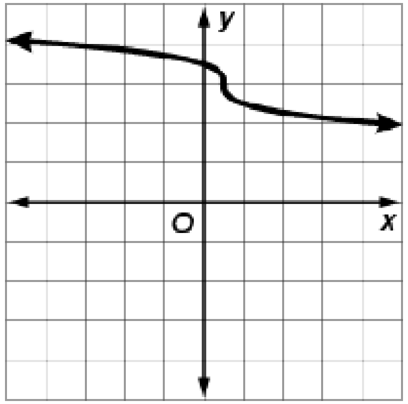
Solution

34. $f(x) = -\frac{1}{2}\sqrt[3]{2x-1}+3$

SOLUTION:

Make a table and graph the function.

x	$f(x)$
−3.5	4
−2.5	≈ 3.91
−1.5	≈ 3.79
−0.5	3.63
0.5	3
1.5	2.37
2.5	≈ 2.21



The domain is all real numbers, or $(-\infty, \infty)$, and the range is all real numbers, or $(-\infty, \infty)$.

Compare to the parent function.

Because the parent function is $y = \sqrt[3]{x}$, the transformed function is $f(x) = a\sqrt[3]{b(x-h)}+k$ where $a = -\frac{1}{2}$, $b = 2$, $h = \frac{1}{2}$, and $k = 3$.

$a < 0$ and $0 < |a| < 1$, so the graph of $y = \sqrt[3]{x}$ is reflected across the x-axis and compressed vertically by a factor of $|a|$, or $\frac{1}{2}$.

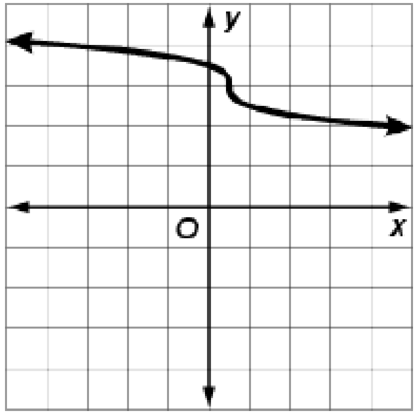
$b > 0$ and $|b| > 1$, so the graph of $y = \sqrt[3]{x}$ is compressed horizontally by a factor of $\left|\frac{1}{b}\right|$, or $\frac{1}{2}$.

$h > 0$, so the graph is then translated right $|h|$ units, or $\frac{1}{2}$ unit.

$k > 0$, so the graph is then translated up k units, or 3 units.

ANSWER:

D = $(-\infty, \infty)$; R = $(-\infty, \infty)$; compressed vertically and horizontally, translated $\frac{1}{2}$ unit right and 3 units up, reflected in the x-axis;



Graph each inequality.

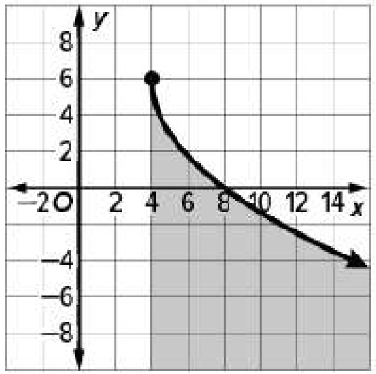
35. $y \le 6 - 3\sqrt{x-4}$

SOLUTION:

Graph the related function.

Graph the boundary $y = 6 - 3\sqrt{x-4}$, using a solid line because the inequality is $\le$.

The domain is $\{x \mid x \ge 4\}$. Because the inequality is less than or equal to, shade the region below the boundary and within the domain.



Select a test point in the shaded region to verify the solution.

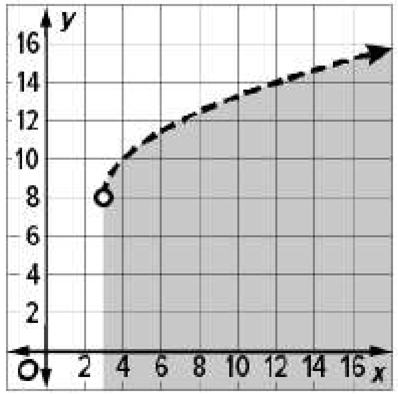
36. $y < \sqrt{4x-12}+8$

SOLUTION:

Graph the related function.

Graph the boundary $y = \sqrt{4x-12}+8$, using a dashed line because the inequality is $<$.

The domain is $\{x \mid x > 3\}$. Because the inequality is less than, shade the region below the boundary and within the domain.



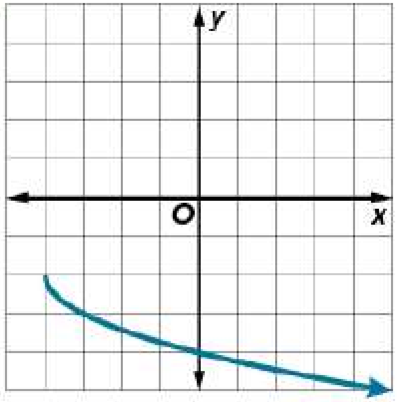
Select a test point in the shaded region to verify the solution.

9	Graph and analyze radical functions	27-39	191

Solution

Write a radical function for each graph.

37.



SOLUTION:

Identify the index.

Because the domain is all real numbers greater than or equal to –4 and range is all real numbers less than or equal to –2, the index is even. This function can be represented by $f(x) = a\sqrt{x-h} + k$.

Identify any transformations.

The function has been translated 4 units left and 2 units down.

Therefore, $h = -4$ and $k = -2$. To find the value of a, use a point as well as the values of h and k.

$f(x) = a\sqrt{x-h} + k$	Square root function
$-4 = a\sqrt{0-(-4)} + (-2)$	$h = -4$, $k = -2$, $(x, f(x)) = (0, -4)$
$-4 = 2a - 2$	Simplify.
$-1 = a$	Simplify.

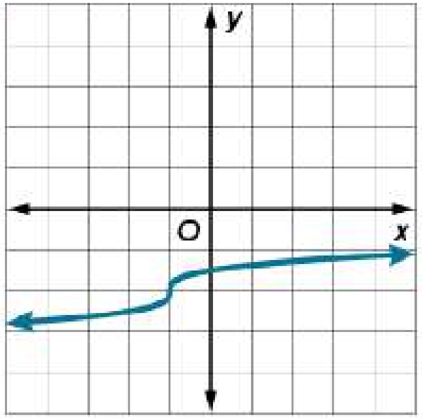
Write the function.

Substitute the values of a, h, and k to write the function. The graph is represented by $f(x) = -\sqrt{x+4} - 2$.

ANSWER:

$f(x) = -\sqrt{x+4} - 2$

38.



SOLUTION:

Identify the index.

Because the domain and range is all real numbers, the index is odd. This function can be represented by $f(x) = a\sqrt[3]{x-h} + k$.

Identify any transformations.

The function has been translated 1 unit left and 2 units down.

Therefore, $h = -1$ and $k = -2$. To find the value of a, use a point as well as the values of h and k.

$f(x) = a\sqrt[3]{x-h} + k$	Cube root function
$-\frac{3}{2} = a\sqrt[3]{0-(-1)} + (-2)$	$h = -1$, $k = -2$, $(x, f(x)) = \left(0, -\frac{3}{2}\right)$
$-\frac{3}{2} = a - 2$	Simplify.
$\frac{1}{2} = a$	Simplify.

Write the function.

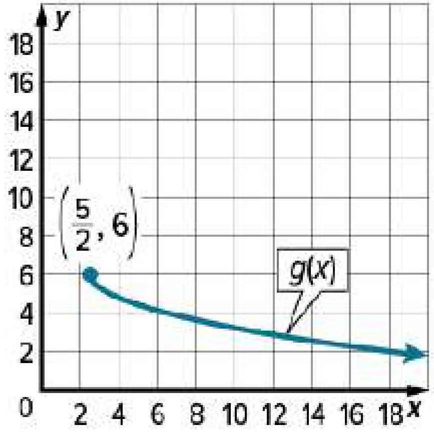
Substitute the values of a, h, and k to write the function. The graph is represented by $f(x) = \frac{1}{2}\sqrt[3]{x+1} - 2$.

ANSWER:

$f(x) = \frac{1}{2}\sqrt[3]{x+1} - 2$

9	Graph and analyze radical functions	27-39	191

39. **STRUCTURE** Consider the function $f(x) = -\sqrt{x+3} + \frac{13}{2}$ and the function $g(x)$ shown in the graph.



a. Determine which function has the greater maximum value. Explain your reasoning.

b. Compare the domains of the two functions.

c. Compare the average rates of change of the two functions over the interval [6, 13].

SOLUTION:

a. The maximum value of $f(x)$ occurs when $x = -3$. So, the maximum value of $f(x)$ is

$$f(-3) = -\sqrt{-3+3} + \frac{13}{2} = \frac{13}{2}.$$

Analyze the graph to find the maximum value of $g(x)$. The maximum value of $g(x)$ occurs when $x = \frac{5}{2}$. So, the maximum value of $g(x)$ is 6.

$f(x)$ has the greater maximum value because its maximum, $\frac{13}{2}$, is greater than 6, the maximum value of $g(x)$.

b. The domain of $f(x)$ is restricted to values for which the radicand is nonnegative.

$x + 3 \geq 0$ Write an inequality using the radicand.
$x \geq -3$ Subtract 3 from each side.

The domain of $f(x)$ is $\{x \mid x \geq -3\}$, since any values less than –3 produce a negative value under the radical.

Analyze the graph to find the domain of $g(x)$.

The domain of $g(x)$ is $\left\{x \mid x \geq \frac{5}{2}\right\}$.

c. Calculate $f(6)$ and $f(13)$.

$f(6) = -\sqrt{6+3} + \frac{13}{2}$ $x = 6$
$= 3.5$ Simplify.

$f(13) = -\sqrt{13+3} + \frac{13}{2}$ $x = 13$
$= 2.5$ Simplify.

Find the average rate of change of $f(x)$ over the interval [6, 13].

$m = \frac{y_2 - y_1}{x_2 - x_1}$ Slope formula
$= \frac{2.5 - 3.5}{13 - 6}$ $(x_1, y_1) = (6, 3.5)$ and $(x_2, y_2) = (13, 2.5)$
$= -\frac{1}{7}$ Simplify.

Find the average rate of change of $g(x)$ over the interval [6, 13].

$m = \frac{y_2 - y_1}{x_2 - x_1}$ Slope formula
$= \frac{3 - 4}{13 - 6}$ $(x_1, y_1) = (6, 4)$ and $(x_2, y_2) = (13, 3)$
$= -\frac{1}{7}$ Simplify.

The average rate of change over the interval is $-\frac{1}{7}$ for $f(x)$. It appears that the rate of change for $g(x)$ is the same.

ANSWER:

a. $f(x)$ has the greater maximum value because its maximum, $\frac{13}{2}$, is greater than 6, the maximum value of $g(x)$.

b. The domain of $f(x)$ is $x \geq -3$, since any values less than –3 produce a negative value under the radical. The domain of $g(x)$ is $x \geq \frac{5}{2}$.

c. The average rate of change over the interval is $-\frac{1}{7}$ for $f(x)$. It appears that the rate of change for $g(x)$ is the same.

10	Add, subtract, multiply, and divide radical expressions	29-38	200

Examples 6 and 7

Simplify.

29. $\dfrac{\sqrt{5a^5}}{\sqrt{b^{13}}}$

30. $\dfrac{\sqrt{7x}}{\sqrt{10x^3}}$

31. $\dfrac{3\sqrt[3]{6x^2}}{3\sqrt[3]{5y}}$

32. $\sqrt[4]{\dfrac{7x^3}{4b^2}}$

33. $\dfrac{6}{\sqrt{3}-\sqrt{2}}$

10	Add, subtract, multiply, and divide radical expressions	29-38	200

34. $\dfrac{\sqrt{2}}{\sqrt{5} - \sqrt{3}}$

35. $\dfrac{9 - 2\sqrt{3}}{\sqrt{3} + 6}$

36. $\dfrac{2\sqrt{2} + 2\sqrt{5}}{\sqrt{5} + \sqrt{2}}$

37. $\dfrac{3\sqrt{7}}{\sqrt{5} - 1}$

38. $\dfrac{7x}{3 - \sqrt{2}}$

10	Add, subtract, multiply, and divide radical expressions	29-38	200

solution method

Lesson 4-5

Operations with Radical Expressions

Learn Rationalizing the Denominator

If a radical expression contains a radical in the denominator, you can rationalize the denominator to simplify the expression. Recall that to rationalize a denominator, you should multiply the numerator and denominator by a quantity so that the radicand has an exact root.

If the denominator is:	Multiply the numerator and denominator by:	Examples
$\sqrt{b}$	$\sqrt{b}$	$\frac{4}{\sqrt{7}} = \frac{4}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}}$ or $\frac{4\sqrt{7}}{7}$
$\sqrt[n]{b^x}$	$\sqrt[n]{b^{n-x}}$	$\frac{3}{\sqrt[5]{2}} = \frac{3}{\sqrt[5]{2}} \cdot \frac{\sqrt[5]{2^4}}{\sqrt[5]{2^4}}$ or $\frac{3\sqrt[5]{16}}{2}$

Binomials of the form $a\sqrt{b} + c\sqrt{d}$ and $a\sqrt{b} - c\sqrt{d}$, where a, b, c, and d are rational numbers, are called **conjugates** of each other. Multiplying the numerator and denominator by the conjugate of the denominator will eliminate the radical from the denominator of the expression.

Example 6 Rationalize the Denominator

Example 7 Use Conjugates to Rationalize the Denominator

11	Solve radical equations	13-20	207

Examples 3 and 4

Solve each equation. Identify any extraneous solutions.

13. $\sqrt{x-15} = 3 - \sqrt{x}$

14. $(5q+1)^{\frac{1}{4}} + 7 = 5$

15. $(3x+7)^{\frac{1}{4}} - 3 = 1$

16. $(3y-2)^{\frac{1}{5}} + 5 = 6$

17. $(4z-1)^{\frac{1}{5}} - 1 = 2$

18. $\sqrt{x-10} = 1 - \sqrt{x}$

19. $\sqrt[6]{y+2} + 9 = 14$

20. $(2x-1)^{\frac{1}{4}} - 2 = 1$

11	Solve radical equations	13-20	207

solution method

Lesson 4-6

Solving Radical Equations

Learn Solving Radical Equations Algebraically

A **radical equation** has a variable in a radicand. When solving a radical equation, the result may be an extraneous solution.

Key Concept • Solving Radical Equations

Step 1 Isolate the radical on one side of the equation.

Step 2 To eliminate the radical, raise each side of the equation to a power equal to the index of the radical.

Step 3 Solve the resulting polynomial equation. Check your results.

Example 3 Identify Extraneous Solutions

Example 4 Solve a Radical Equation

12	Graph growth and decay functions	1-13	221

Example 1

Graph each function. Find the domain, range, *y*-intercept, asymptote, and end behavior.

1. $f(x) = 3^x$

2. $f(x) = 5^x$

12	Graph growth and decay functions	1-13	221

Example 1

Graph each function. Find the domain, range, *y*-intercept, asymptote, and end behavior.

3. $f(x) = 1.5^x$

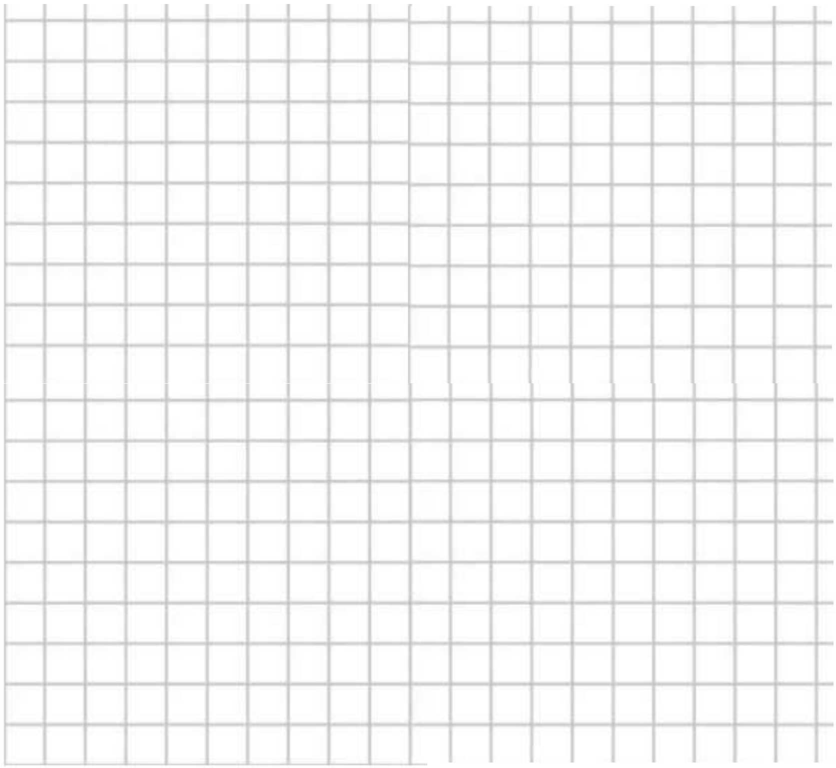
4. $f(x) = \left(\frac{5}{2}\right)^x$

12	Graph growth and decay functions	1-13	221

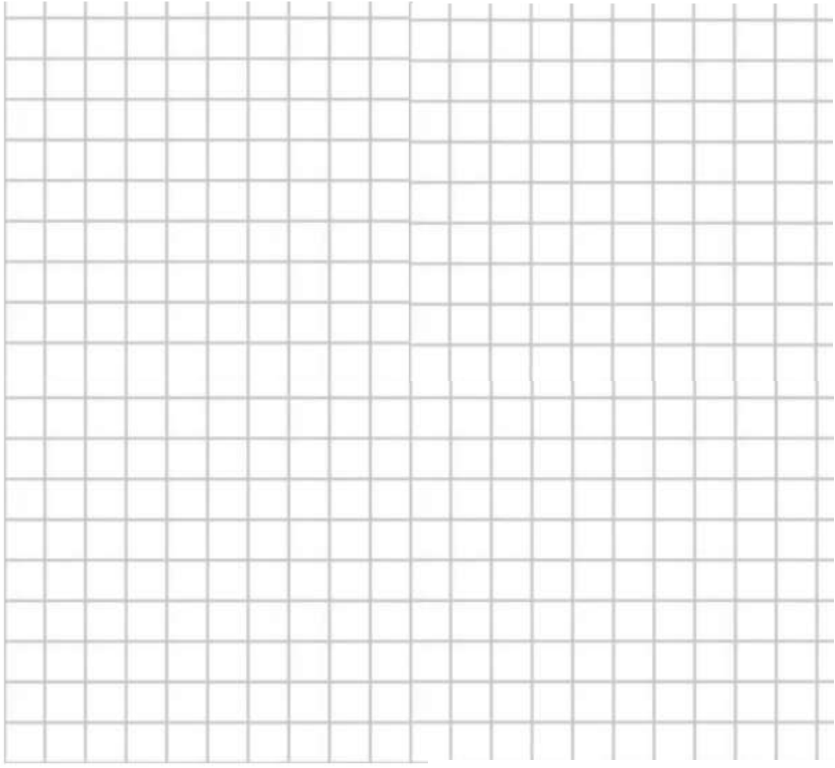
Example 2

Graph each function.

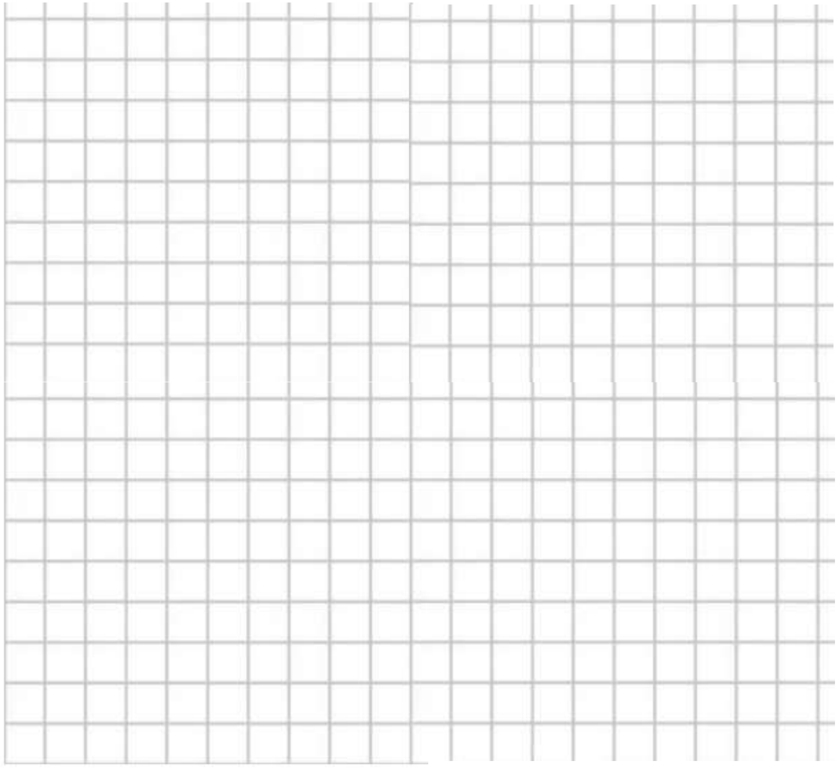
5. $f(x) = 2(3)^x$



6. $f(x) = -2(4)^x$



7. $f(x) = 4^{x+1} - 5$



12	Graph growth and decay functions	1-13	221

Example 2

Graph each function.

8. $f(x) = 3^{2x} + 1$

9. $f(x) = -0.4(3)^{x+2} + 4$

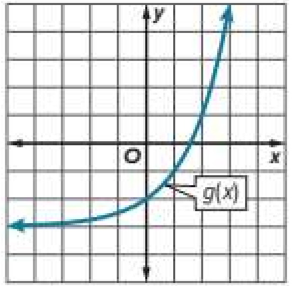
10. $f(x) = 1.5(2)^{x} + 6$

12	Graph growth and decay functions	1-13	221

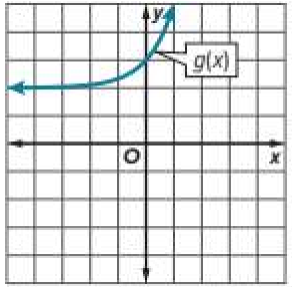
Example 3

Identify the value of k and write a function $g(x)$ for each graph as it relates to $f(x)$.

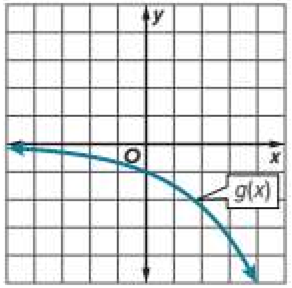
11. $f(x) = 2^x$; $g(x) = f(x) + k$



12. $f(x) = 3^x$; $g(x) = f(x) + k$



13. $f(x) = \left(\frac{3}{2}\right)^x$; $g(x) = k \cdot f(x)$



12	Graph growth and decay functions	1-13	221

Solution

solution method

Lesson 5-1

Graphing Exponential Functions

Learn Graphing Exponential Growth Functions

In an **exponential function,** the independent variable is an exponent. An exponential function has the form $f(x) = b^x$, where the base b is a constant and the independent variable x is the exponent. For an exponential growth function, $b > 1$. **Exponential growth** occurs when an initial amount increases by the same percent over a given period of time.

Graphs of exponential functions have asymptotes. An **asymptote** is a line that a graph approaches.

Example 1 Graph Exponential Growth Functions

Example 2 Graph Transformations of Exponential Growth Functions

Example 3 Analyze Graphs of Exponential Functions

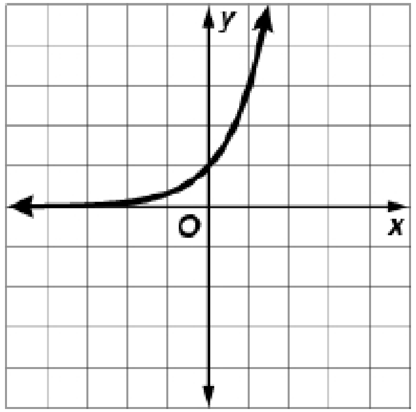
Graph each function. Find the domain, range, y-intercept, asymptote, and end behavior.

1. $f(x) = 3^x$

SOLUTION:

Make a table of values. Then plot the points and sketch the graph.

x	$f(x) = 3^x$
−3	≈ 0.037
−2	≈ 0.111
−1	≈ 0.333
0	1
1	3
2	9



domain: all real numbers

range: all positive real numbers

y-intercept: (0, 1)

asymptote: $y = 0$

end behavior: As $x \to -\infty$, $f(x) \to 0$ and as $x \to \infty$, $f(x) \to \infty$.

ANSWER:

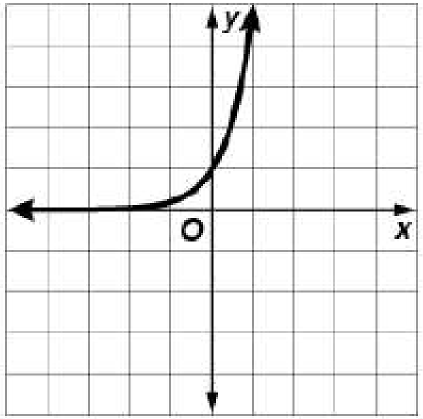
Domain: all real numbers; Range: all positive real numbers; y-intercept: (0, 1); Asymptote: $y = 0$; End Behavior: as $x \to -\infty$, $f(x) \to 0$ and as $x \to \infty$, $f(x) \to \infty$

2. $f(x) = 5^x$

SOLUTION:

Make a table of values. Then plot the points and sketch the graph.

x	$f(x) = 5^x$
−3	0.008
−2	0.04
−1	0.2
0	1
1	5
2	25



domain: all real numbers

range: all positive real numbers

y-intercept: (0, 1)

asymptote: $y = 0$

end behavior: As $x \to -\infty$, $f(x) \to 0$ and as $x \to \infty$, $f(x) \to \infty$.

ANSWER:

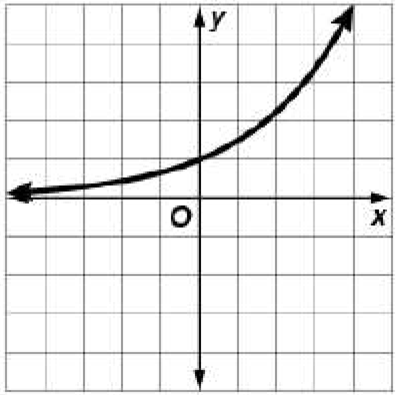
Domain: all real numbers; Range: all positive real numbers; y-intercept: (0, 1); Asymptote: $y = 0$; End Behavior: as $x \to -\infty$, $f(x) \to 0$ and as $x \to \infty$, $f(x) \to \infty$

3. $f(x) = 1.5^x$

SOLUTION:

Make a table of values. Then plot the points and sketch the graph.

x	$f(x) = 1.5^x$
−3	≈ 0.296
−2	≈ 0.444
−1	≈ 0.667
0	1
1	1.5
2	2.25



domain: all real numbers

range: all positive real numbers

y-intercept: (0, 1)

asymptote: $y = 0$

end behavior: As $x \to -\infty$, $f(x) \to 0$ and as $x \to \infty$, $f(x) \to \infty$.

ANSWER:

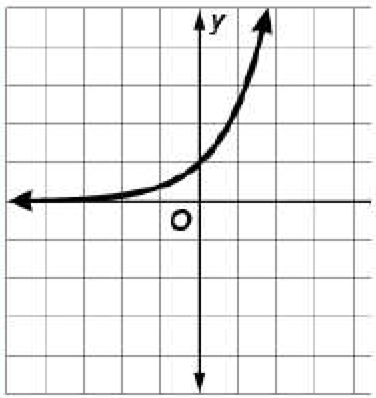
Domain: all real numbers; Range: all positive real numbers; y-intercept: (0, 1); Asymptote: $y = 0$; End Behavior: as $x \to -\infty$, $f(x) \to 0$ and as $x \to \infty$, $f(x) \to \infty$

4. $f(x) = \left(\frac{5}{2}\right)^x$

SOLUTION:

Make a table of values. Then plot the points and ske the graph.

x	
−3	0.064
−2	0.16
−1	0.4
0	1
1	2.5
2	6.25



domain: all real numbers

range: all positive real numbers

y-intercept: (0, 1)

asymptote: $y = 0$

end behavior: As $x \to -\infty$, $f(x) \to 0$ and as $x \to \infty$, $f(x) \to \infty$.

ANSWER:

Domain: all real numbers; Range: all positive real numbers; y-intercept: (0, 1); Asymptote: $y = 0$; End Behavior: as $x \to -\infty$, $f(x) \to 0$ and as $x \to \infty$, $f(x) \to \infty$

Graph each function.

5. $f(x) = 2(3)^x$

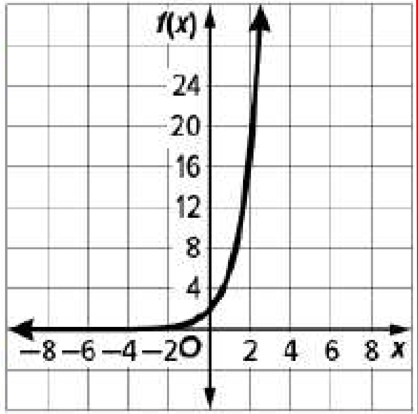
SOLUTION:

Transform the graph of $f(x) = 3^x$.

$a = 2$, so stretch the graph vertically.

$h = 0$, so do not translate the graph left or right.

$k = 0$, so do not translate the graph up or down.



6. $f(x) = -2(4)^x$

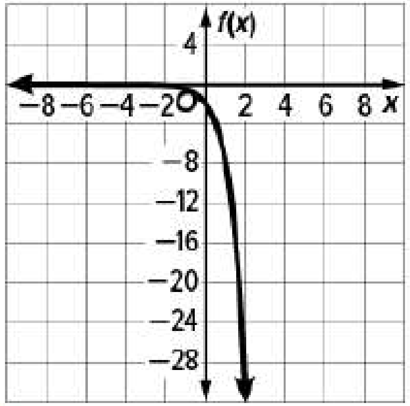
SOLUTION:

Transform the graph of $f(x) = 4^x$.

$a = -2$, so reflect the graph in in the x-axis and stretch vertically.

$h = 0$, so do not translate the graph left or right.

$k = 0$, so do not translate the graph up or down.



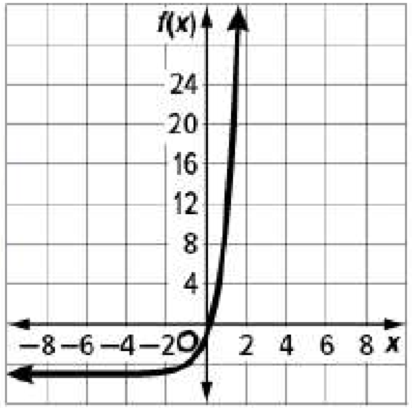
12	Graph growth and decay functions	1-13	221

Solution

7. $f(x) = 4^{x+1} - 5$

SOLUTION:

Transform the graph of $f(x) = 4^x$.



$a = 1$, so do not stretch or compress the graph vertically.

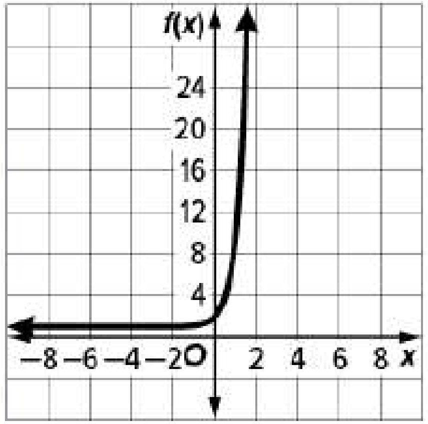
$h = -1$, so translate the graph 1 unit left.

$k = -5$, so translate the graph 5 units down.

8. $f(x) = 3^{2x} + 1$

SOLUTION:

Transform the graph of $f(x) = 3^{2x}$.



$a = 1$, so do not stretch or compress the graph vertically.

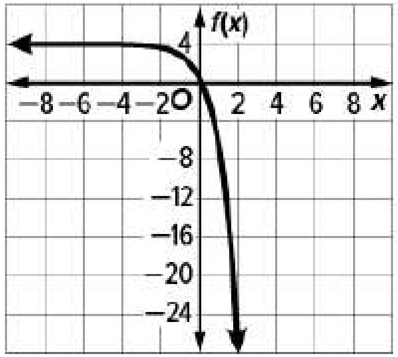
$h = 0$, so do not translate the graph left or right.

$k = 1$, so translate the graph 1 unit up.

9. $f(x) = -0.4(3)^{x+2} + 4$

SOLUTION:

Transform the graph of $f(x) = 3^x$.



$a = -0.4$, so reflect the graph in in the x-axis and compress vertically.

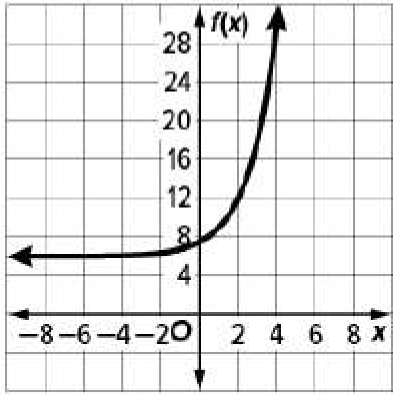
$h = -2$, so translate the graph 2 units left.

$k = 4$, so translate the graph 4 units up.

10. $f(x) = 1.5(2)^x + 6$

SOLUTION:

Transform the graph of $f(x) = 2^x$.



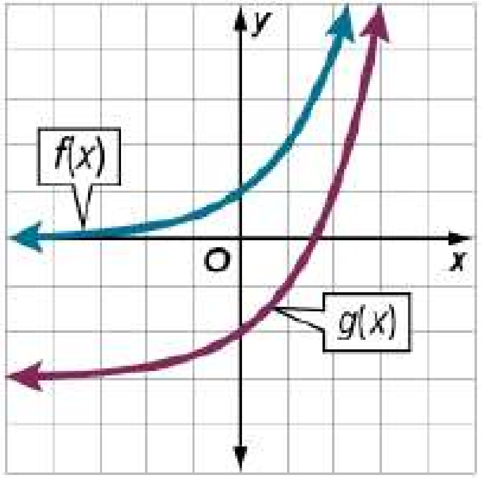
$a = 1.5$, so stretch the graph vertically.

$h = 0$, so do not translate the graph left or right.

$k = 6$, so translate the graph 6 units up.

Identify the value of k and write a function $g(x)$ for each graph as it relates to $f(x)$.

11. $f(x) = 2^x$; $g(x) = f(x) + k$



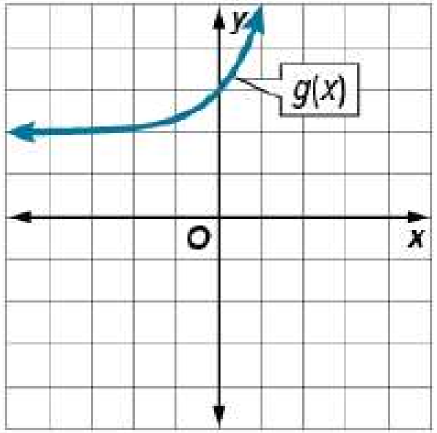
SOLUTION:

The graph has been translated 3 units down, so $k = -3$ and the function is $g(x) = 2^x - 3$.

ANSWER:

$k = -3$; $g(x) = 2^x - 3$

12. $f(x) = 3^x$; $g(x) = f(x) + k$



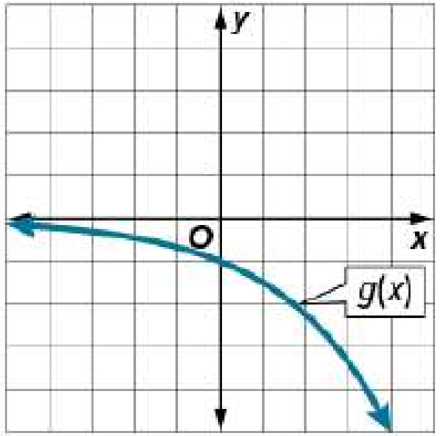
SOLUTION:

The graph has been translated 2 units up, so $k = 2$ and the function is $g(x) = 3^x + 2$.

ANSWER:

$k = 2$; $g(x) = 3^x + 2$

13. $f(x) = \left(\frac{3}{2}\right)^x$; $g(x) = k \cdot f(x)$



SOLUTION:

The graph has been reflected in the x-axis, so $a = -1$ and the function is $g(x) = -\left(\frac{3}{2}\right)^x$.

ANSWER:

$k = -1$; $g(x) = -\left(\frac{3}{2}\right)^x$

13	Solve exponential equations	1-6	229

Example 1

Solve each equation.

1. $25^{2x+3} = 25^{5x-9}$

2. $9^{8x-4} = 81^{3x+6}$

3. $4^{x-5} = 16^{2x-31}$

4. $4^{3x-3} = 8^{4x-4}$

5. $9^{-x+5} = 27^{6x-10}$

6. $125^{3x-4} = 25^{4x+2}$

13	Solve exponential equations	1-6	229

Solution

solution method

Lesson 5-2

Solving Exponential Equations and Inequalities

Learn Solving Exponential Equations

In an **exponential equation**, the independent variable is an exponent.

Key Concept • Property of Equality for Exponential Equations

If $b > 0$ and $b \neq 1$, then $b^x = b^y$ if and only if $x = y$.

Exponential equations can be solved algebraically or by graphing a system of equations based on the equation.

Equations of exponential functions can be used to calculate compound interest. **Compound interest** is paid on the principal of an investment and any previously earned interest.

Key Concept • Compound Interest

You can calculate compound interest using the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$, where A is the amount in the account after t years, P is the principal amount invested, r is the annual interest rate, and n is the number of compounding periods each year.

Example 1 Solve Exponential Equations Algebraically

Solve each equation

1. $25^{2x+3} = 25^{5x-9}$

SOLUTION:

$25^{2x+3} = 25^{5x-9}$ Original equa
$2x + 3 = 5x - 9$ Property of E
$2x + 12 = 5x$ Add 9 to eacl
$12 = 3x$ Subtract 2x f
$4 = x$ Divide each

ANSWER:

4

2. $9^{8x-4} = 81^{2x+6}$

SOLUTION:

$9^{8x-4} = 81^{3x+6}$ Original equa
$9^{8x-4} = (9^2)^{3x+6}$ Rewrite 81 a
$9^{8x-4} = 9^{6x+12}$ Power of a P
$8x - 4 = 6x + 12$ Property of E
$8x = 6x + 16$ Add 4 to eacl
$2x = 16$ Subtract 6x f
$x = 8$ Divide each

ANSWER:

8

3. $4^{x-5} = 16^{2x-31}$

SOLUTION:

$4^{x-5} = 16^{2x-31}$ Original
$4^{x-5} = (4^2)^{2x-31}$ Rewrite
$4^{x-5} = 4^{4x-62}$ Power of
$x - 5 = 4x - 62$ Property
$x + 57 = 4x$ Add 62 t
$57 = 3x$ Subtract
$19 = x$ Divide e

ANSWER:

19

4. $4^{3x-3} = 8^{4x-4}$

SOLUTION:

$4^{3x-3} = 8^{4x-4}$ Original
$(4^2)^{3x-3} = (2^3)^{4x-4}$ Rewrite
$4^{6x-6} = 4^{12x-12}$ Power o
$6x - 6 = 12x - 12$ Property
$6x + 6 = 12x$ Add 12
$6 = 6x$ Subtract
$1 = x$ Divide

ANSWER:

1

5. $9^{-x+5} = 27^{6x-10}$

SOLUTION:

$9^{-x+5} = 27^{6x-10}$ Original equa
$(3^2)^{-x+5} = (3^3)^{6x-10}$ Rewrite 9 as 3
$3^{-2x+10} = 3^{18x-30}$ Power of a Po
$-2x + 10 = 18x - 30$ Property of E
$-2x + 40 = 18x$ Add 30 to eac
$40 = 20x$ Add 2x to eac
$2 = x$ Divide each s

ANSWER:

2

6. $125^{3x-4} = 25^{4x+2}$

SOLUTION:

$125^{3x-4} = 25^{4x+2}$ Original equat
$(5^3)^{3x-4} = (5^2)^{4x+2}$ Rewrite 125 as
$5^{9x-12} = 5^{8x+4}$ Power of a Pow
$9x - 12 = 8x + 4$ Property of Eq
$9x = 8x + 16$ Add 12 to eacl
$x = 16$ Subtract 8x fr

ANSWER:

16

14	Evaluate expressions involving the natural base and natural logarithm	10-15	237

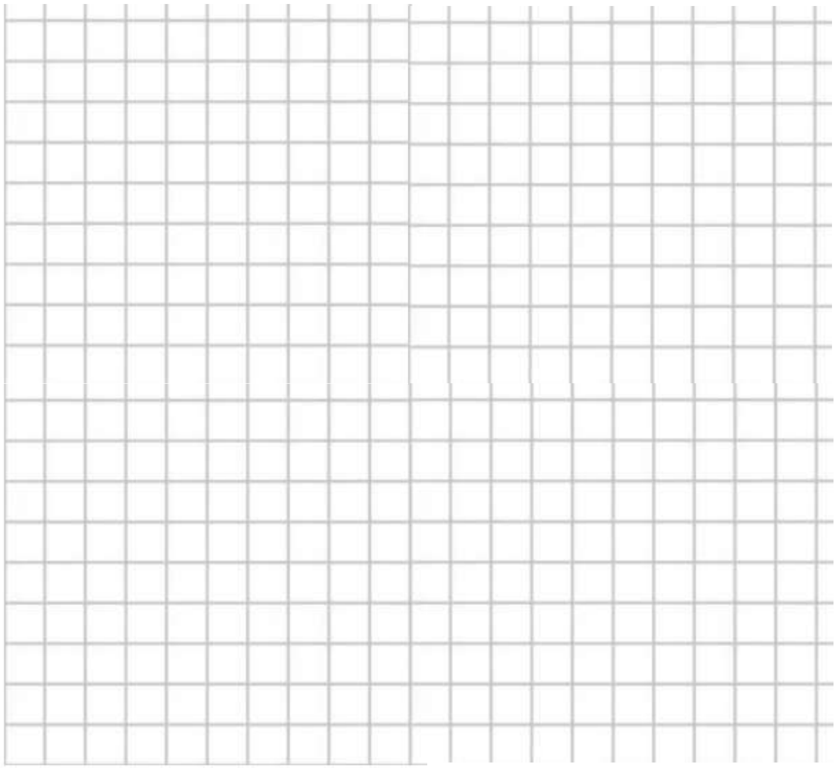
Example 2

10. Consider the function $f(x) = 3e^{x-1} + 3$.

a. Graph the function.

b. Determine domain and range.

c. Find the average rate of change over the interval $[-5, -2]$.

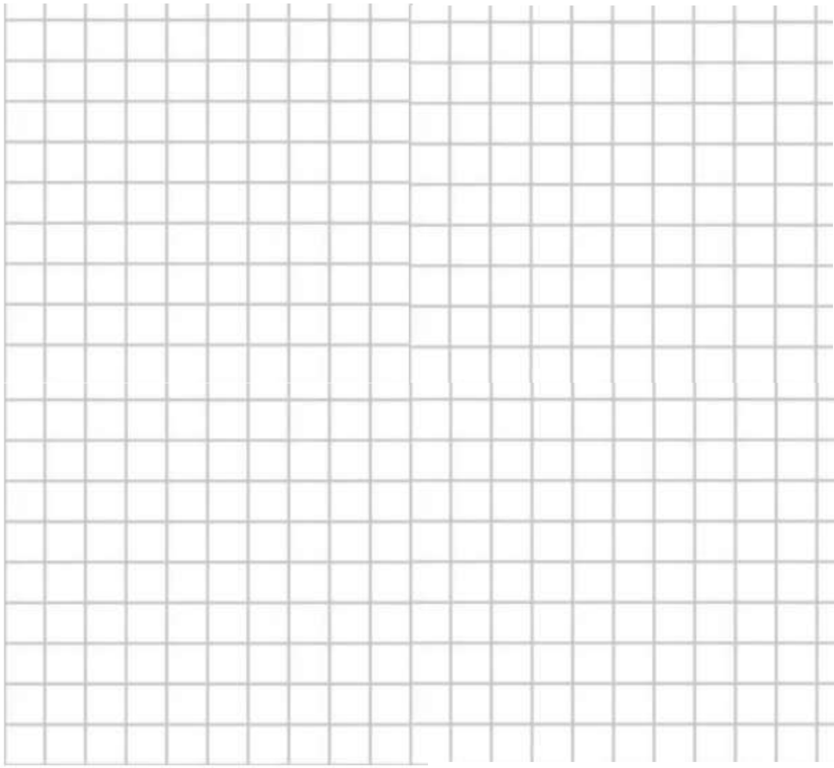


11. Consider the function $f(x) = 4e^{2x} - 1$.

a. Graph the function.

b. Determine domain and range.

c. Find the average rate of change over the interval $[-3, -1]$.



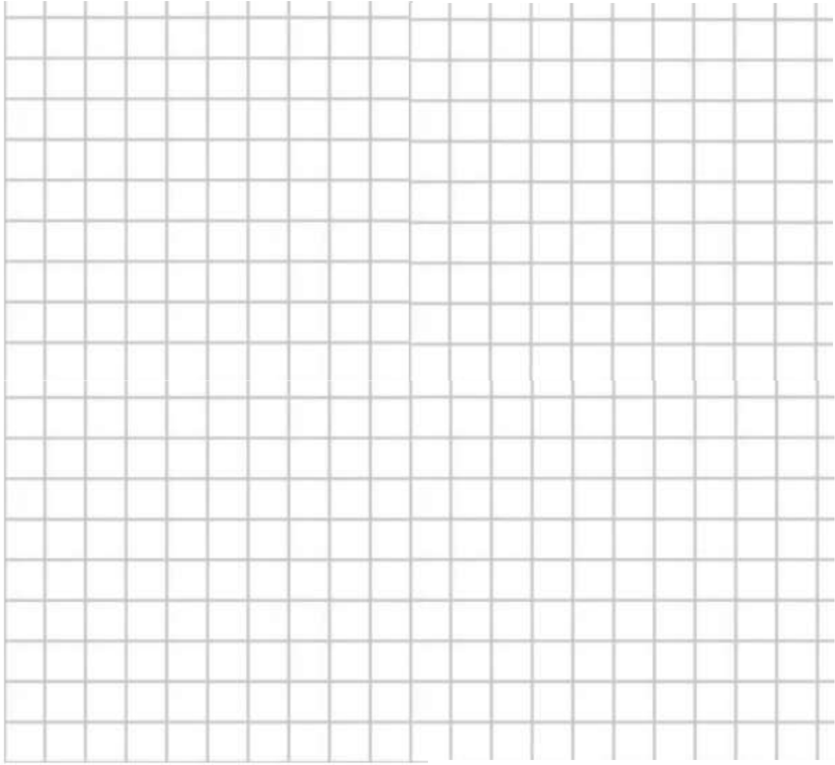
14	Evaluate expressions involving the natural base and natural logarithm	10-15	237

12. Consider the function $f(x) = -2e^{x+3} + 2$.

a. Graph the function.

b. Determine domain and range.

c. Find the average rate of change over the interval $[-7, -4]$.



14	Evaluate expressions involving the natural base and natural logarithm	10-15	237

Example 3

13. COMPOUND INTEREST Ryan invested $5000 in an account that grows continuously at an annual rate of 2.5%.

a. Write the function that represents the situation, where A is the value of Ryan's investment after t years.

b. What will Ryan's investment will be worth after 7 years?

14. SAVINGS Jariah invested $6500 in a savings account that grows continuously at an annual rate of 3.25%.

a. Write the function that represents the situation, where A is the value of Jariah's investment after t years.

b. What will Jariah's investment will be worth after 18 years?

15. INVESTMENTS Marcella invested $12,750 in a company. Her investment has been growing continuously at an annual rate of 5.5%.

a. Write the function that represents the situation, where A is the value of Marcella's investment after t years.

b. What will Marcella's investment will be worth after 9 years?

14	Evaluate expressions involving the natural base and natural logarithm	10-15	237

Solution

solution method

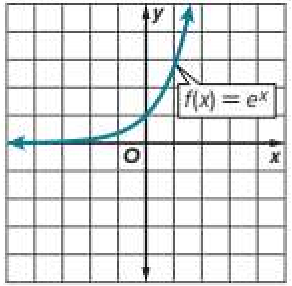
Lesson 5-3

Special Exponential Functions

Learn Exponential Functions with Base *e*

The constant **e** has certain mathematical properties that make it a convenient base for exponential functions. *e* is the irrational number that $\left(1+\frac{1}{n}\right)^n$ approaches as *n* approaches ∞. This value is approximately equal to 2.7182818...

Graphs of exponential functions with base *e* display the same general characteristics as other exponential functions.



- $f(x)$ approaches an asymptote of $y = 0$ as x approaches $-\infty$.
- $f(x)$ approaches ∞ as x approaches ∞.
- The y-intercept is 1.

You can calculate continuously compounding interest by using the formula $A = Pe^{rt}$, where *A* is the amount in the account after *t* years, *P* is the principal amount invested, and *r* is the annual interest rate.

Example 2 Graph Functions with Base *e*

Example 3 Apply Functions with Base *e*

10. Consider the function $f(x) = 3e^{x-1} + 3$.

a. Graph the function.

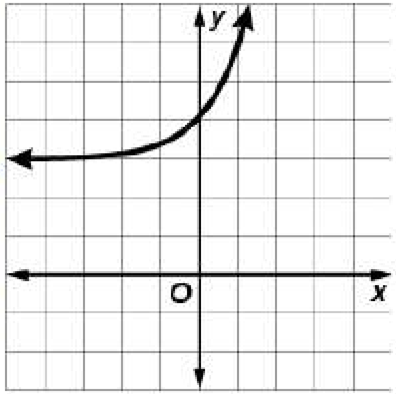
b. Determine domain and range.

c. Find the average rate of change over the interval [–5, –2].

SOLUTION:

a.

The function $f(x) = 3e^{x-1} + 3$ represents a transformation of the graph of $f(x) = e^x$.



$a = 3$, so the graph is stretched vertically.

$h = 1$, so the graph is translated 1 unit right.

$k = 3$, so the graph is translated 3 units up.

b. The domain is all real numbers. The range is all real numbers greater than 3.

c. Based on the graph, the graph from –5 to –2 appears approximately horizontal. So, the average rate of change should be close to 0.

$\frac{f(-2)-f(-5)}{-2-(-5)} \approx \frac{3.149-3.007}{3}$ Evaluate $f(-2)$ and $f(-5)$.

$\approx \frac{0.142}{3}$ or 0.047 Simplify.

11. Consider the function $f(x) = 4e^{2x} - 1$.

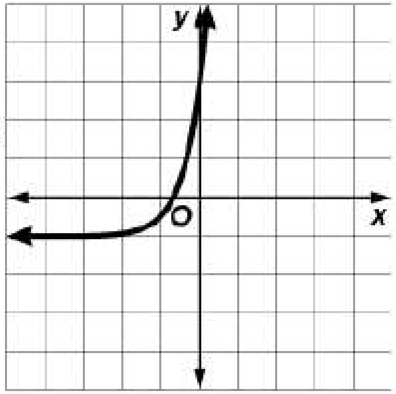
a. Graph the function.

b. Determine domain and range.

c. Find the average rate of change over the interval [–3, –1].

SOLUTION:

The function $f(x) = 4e^{2x} - 1$ represents a transformation of the graph of $f(x) = e^{2x}$.



$a = 4$, so the graph is stretched vertically.

$h = 0$, so the graph is not translated right or left.

$k = -1$, so the graph is translated 1 unit down.

b. The domain is all real numbers. The range is all rea numbers greater than –1.

c. Based on the graph, the graph from –3 to –1 appears approximately horizontal. So, the average rate of change should be close to 0.

$\frac{f(-1)-f(-3)}{-1-(-3)} \approx \frac{-0.459-(-0.990)}{2}$ Evaluate $f(-1)$ and $f(-3)$.

$\approx \frac{0.531}{2}$ or 0.226 Simplify.

12. Consider the function $f(x) = -2e^{x+3} + 2$.

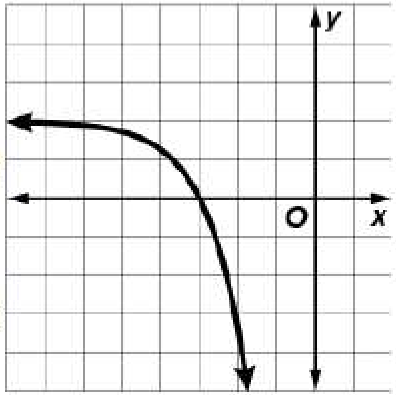
a. Graph the function.

b. Determine domain and range.

c. Find the average rate of change over the interval [–7, –4].

SOLUTION:

a.



The function $f(x) = -2e^{x+3} + 2$ represents a transformation of the graph of $f(x) = e^x$.

$a = -2$, so the graph is reflected in the x-axis and stretched vertically.

$h = -3$, so the graph is translated 3 units left.

$k = 2$, so the graph is translated 2 units up.

b. The domain is all real numbers. The range is all rea numbers less than 2.

c. Based on the graph, the graph from –7 to –4 appears approximately horizontal. So, the average rate of change should be close to 0.

$\frac{f(-4)-f(-7)}{-4-(-7)} \approx \frac{1.264-1.963}{3}$ Evaluate $f(-4)$ and $f(-7)$.

$\approx \frac{-0.699}{3}$ or -0.233 Simplify.

13. **COMPOUND INTEREST** Ryan invested $5000 in an account that grows continuously at an annual rate of 2.5%.

a. Write the function that represents the situation, where A is the value of Ryan's investment after t years.

b. What will Ryan's investment will be worth after 7 years?

SOLUTION:

a. To write a function that represents the situation, use the formula for continuous exponential growth.

$A = Pe^{rt}$ Continuous Compounding Formula

$A = 5000e^{0.025t}$ $P = 5000$, and $r = 0.025$

b. To find what Ryan's investment will be worth after 7 years, let $t = 7$.

$A = 5000e^{0.025t}$ Original function

$= 5000e^{0.025(7)}$ $t = 7$

$= 5956.23$ Simplify.

After 7 years, Ryan's investment will be worth $5956.23.

ANSWER:

a. $A = 5000e^{0.025t}$
b. $5956.23

14	Evaluate expressions involving the natural base and natural logarithm	10-15	237

Solution

14. **SAVINGS** Jariah invested $6500 in a savings account that grows continuously at an annual rate of 3.25%.

a. Write the function that represents the situation, where A is the value of Jariah's investment after t years.

b. What will Jariah's investment will be worth after 18 years?

SOLUTION:

a. To write a function that represents the situation, use the formula for continuous exponential growth.

$A = Pe^{rt}$ Continuous Compounding Formula
$= 6500e^{0.0325t}$ $P = 6500$, and $r = 0.0325$

b. To find what Jariah's investment will be worth after 18 years, let $t = 18$.

$A = 6500e^{0.0325t}$ Original function
$= 6500e^{0.0325(18)}$ $t = 18$
$= 11{,}667.44$ Simplify.

After 18 years, Jariah's investment will be worth $11,667.44.

ANSWER:

a. $A = 6500e^{0.0325t}$
b. $11,667.44

15. **INVESTMENTS** Marcella invested $12,750 in a company. Her investment has been growing continuously at an annual rate of 5.5%.

a. Write the function that represents the situation, where A is the value of Marcella's investment after t years.

b. What will Marcella's investment will be worth after 9 years?

SOLUTION:

a. To write a function that represents the situation, use the formula for continuous exponential growth.

$A = Pe^{rt}$ Continuous Compounding Formula
$= 12{,}750e^{0.055t}$ $P = 12{,}750$, and $r = 0.055$

b. To find what Marcella's investment will be worth after 9 years, let $t = 9$.

$A = 12{,}750e^{0.055t}$ Original function
$= 12{,}750e^{0.055(9)}$ $t = 9$
$= 20{,}916.35$ Simplify.

After 9 years, Marcella's investment will be worth $20,916.35.

ANSWER:

a. $A = 12{,}750e^{0.055t}$
b. $20,916.35

15	Simplify rational expressions	1-16	315

Example 1

Simplify each expression, and state when the original expression is undefined.

1. $\dfrac{x(x-3)(x+6)}{x^2+x-12}$

2. $\dfrac{y^2(y^2+3y+2)}{2y(y-4)(y+2)}$

3. $\dfrac{(x^2-9)(x^2-z^2)}{4(x+z)(x-3)}$

4. $\dfrac{(x^2-16x+64)(x+2)}{(x^2-64)(x^2-6x-16)}$

5. $\dfrac{x^2(x+2)(x-4)}{6x(x^2+x-20)}$

6. $\dfrac{3y(y-8)(y^2+2y-24)}{15y^2(y^2-12y+32)}$

15	Simplify rational expressions	1-16	315

Example 2

Simplify each expression.

7. $\dfrac{x^2 - 5x - 14}{28 + 3x - x^2}$

8. $\dfrac{9x^2 - x^3}{x^2 - 3x - 54}$

9. $\dfrac{(x - 4)(x^2 + 2x - 48)}{(36 - x^2)(x^2 + 4x - 32)}$

10. $\dfrac{16 - c^2}{c^2 + c - 20}$

15	Simplify rational expressions	1-16	315

Example 3

Simplify each expression.

11. $\frac{3ac^3f^3}{8a^2bcf^4} \cdot \frac{12ab^2c}{18ab^3c^2f}$

12. $\frac{14xy^2z^3}{21w^4x^2z} \cdot \frac{7wxyz}{12w^2y^3z}$

13. $\frac{64a^2b^5}{35b^2c^3f^4} \div \frac{12a^4b^3c}{70abcf^2}$

14. $\frac{9x^2yz}{5z^4} \div \frac{12x^4y^2}{50xy^4z^2}$

15. $\frac{15a^2b^2}{21ac} \cdot \frac{14a^4c^2}{6ab^3}$

16. $\frac{14c^2f^5}{9a^2} \div \frac{35cf^4}{18ab^3}$

15	Simplify rational expressions	1-16	315

solution method

Lesson 7-1

Multiplying and Dividing Rational Expressions

Learn Simplifying Rational Expressions

A **rational expression** is a ratio of two polynomial expressions.

Because variables in algebra often represent real numbers, operations with rational numbers and rational expressions are similar. For example, when you write a fraction in simplest form, you divide the numerator and denominator by the greatest common factor (GCF).

$$\frac{35}{40} = \frac{\cancel{5} \cdot 7}{\cancel{5} \cdot 8} = \frac{7}{8} \qquad \text{GCF} = 5$$

You use the same process to simplify a rational expression.

$$\frac{x^2 + 7x + 10}{x^2 - x - 6} = \frac{(x+5)\cancel{(x+2)}}{(x-3)\cancel{(x+2)}} = \frac{(x+5)}{(x-3)} \qquad \text{GCF} = x + 2$$

Sometimes, you can also factor out −1 in the numerator or denominator to help simplify a rational expression.

Example 2 Simplify by Using −1

Learn Multiplying and Dividing Rational Expressions

The method for multiplying and dividing fractions also works with rational expressions.

Key Concept • Multiplying Rational Expressions

Words: To multiply rational expressions, multiply the numerators and the denominators.

Symbols: For all rational expressions $\frac{a}{b}$ and $\frac{c}{d}$ with $b \neq 0$ and $d \neq 0$, $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

Key Concept • Dividing Rational Expressions

Words: To divide rational expressions, multiply the dividend by the reciprocal of the divisor.

Symbols: For all rational expressions $\frac{a}{b}$ and $\frac{c}{d}$ with $b \neq 0$, $c \neq 0$, and $d \neq 0$, $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$.

A **complex fraction** is a rational expression with a numerator and/or denominator that is also a rational expression. To simplify a complex fraction, first rewrite it as a division expression.

Example 3 Multiply and Divide Rational Expressions

Simplify each expression, and state when the original expression is undefined.

1. $\dfrac{x(x-3)(x+6)}{x^2+x-12}$

SOLUTION:

$\dfrac{x(x-3)(x+6)}{x^2+x-12} = \dfrac{x(x-3)(x+6)}{(x+4)(x-3)}$ Factor the denominator.

$= \dfrac{x(x+6)}{(x+4)} \cdot \dfrac{(x-3)}{(x-3)}$ Eliminate common factors.

$= \dfrac{x(x+6)}{(x+4)}$ Simplify.

By the Zero Product Property, the expression is undefined when $(x + 4) = 0$ or $x = -4$, and when $(x - 3) = 0$ or $x = 3$.

ANSWER:

$\dfrac{x(x+6)}{x+4}$; $x = -4, 3$

2. $\dfrac{y^2(y^2+3y+2)}{2y(y-4)(y+2)}$

SOLUTION:

$\dfrac{y^2(y^2+3y+2)}{2y(y-4)(y+2)} = \dfrac{y^2(y+1)(y+2)}{2y(y-4)(y+2)}$ Factor the numerator.

$= \dfrac{y(y+1)}{2(y-4)} \cdot \dfrac{(y+2)}{(y+2)}$ Eliminate common factors.

$= \dfrac{y(y+1)}{2(y-4)}$ Simplify.

By the Zero Product Property, the expression is undefined when $y = 0$, $(y - 4) = 0$ or $y = 4$, and when $(y + 2) = 0$ or $y = -2$.

ANSWER:

$\dfrac{y(y+1)}{2(y-4)}$; $y = -2, 0, 4$

3. $\dfrac{(x^2-9)(x^2-z^2)}{4(x+z)(x-3)}$

SOLUTION:

$\dfrac{(x^2-9)(x^2-z^2)}{4(x+z)(x-3)} = \dfrac{(x-3)(x+3)(x-z)(x+z)}{4(x+z)(x-3)}$ Factor the numerator.

$= \dfrac{(x+3)(x-z)}{4} \cdot \dfrac{(x-3)}{(x-3)} \cdot \dfrac{(x+z)}{(x+z)}$ Eliminate common factors.

$= \dfrac{(x+3)(x-z)}{4}$ Simplify.

By the Zero Product Property, the expression is undefined when $(x - 3) = 0$ or $x = 3$, and when $(x + z) = 0$ or $x = -z$.

ANSWER:

$\dfrac{(x+3)(x-z)}{4}$; $x = -z, 3$

4. $\dfrac{(x^2-16x+64)(x+2)}{(x^2-64)(x^2-6x-16)}$

SOLUTION:

$\dfrac{(x^2-16x+64)(x+2)}{(x^2-64)(x^2-6x-16)} = \dfrac{(x-8)(x-8)(x+2)}{(x+8)(x-8)(x-8)(x+2)}$ Factor the denominator and numerator.

$= \dfrac{1}{(x+8)} \cdot \dfrac{(x-8)}{(x-8)} \cdot \dfrac{(x-8)}{(x-8)} \cdot \dfrac{(x+2)}{(x+2)}$ Eliminate common factors.

$= \dfrac{1}{(x+8)}$ Simplify.

By the Zero Product Property, the expression is undefined when $(x^2 - 64) = 0$ or $x = -8$ and 8, and when $(x + 2) = 0$ or $x = -2$.

ANSWER:

$\dfrac{1}{x+8}$; $x = -8, -2, 8$

5. $\dfrac{x^2(x+2)(x-4)}{6x(x^2+x-20)}$

SOLUTION:

$\dfrac{x^2(x+2)(x-4)}{6x(x^2+x-20)} = \dfrac{x^2(x+2)(x-4)}{6x(x+5)(x-4)}$ Factor the denominator.

$= \dfrac{x(x+2)}{6(x+5)} \cdot \dfrac{x}{x} \cdot \dfrac{(x-4)}{(x-4)}$ Eliminate common factors.

$= \dfrac{x(x+2)}{6(x+5)}$ Simplify.

By the Zero Product Property, the expression is undefined when $6x = 0$ or when $x = 0$, and when $(x + 5) = 0$ or $x = -5$, and when $(x - 4) = 0$ or $x = 4$.

ANSWER:

$\dfrac{x(x+2)}{6(x+5)}$; $x = -5, 0, 4$

6. $\dfrac{3y(y-8)(y^2+2y-24)}{15y^2(y^2-12y+32)}$

SOLUTION:

$\dfrac{3y(y-8)(y^2+2y-24)}{15y^2(y^2-12y+32)} = \dfrac{3y(y-8)(y-4)(y+6)}{15y^2(y-8)(y-4)}$ Factor the numerator and denominator.

$= \dfrac{3y}{15y^2} \cdot \dfrac{(y-8)}{(y-8)} \cdot \dfrac{(y-4)}{(y-4)} \cdot \dfrac{(y+6)}{1}$ Eliminate common factors.

$= \dfrac{(y+6)}{5y}$ Simplify.

By the Zero Product Property, the expression is undefined when $15y^2 = 0$ or when $y = 0$, and when $(y - 8) = 0$ or $y = 8$, and when $(y - 4) = 0$ or $y = 4$.

ANSWER:

$\dfrac{(y+6)}{5y}$; $x = 0, 4, 8$

Simplify each expression.

7. $\dfrac{x^2-5x-14}{28+3x-x^2}$

SOLUTION:

$\dfrac{x^2-5x-14}{28+3x-x^2} = \dfrac{(x-7)(x+2)}{(7-x)(4+x)}$ Factor.

$= \dfrac{(x-7)(x+2)}{(-1)(x-7)(x+4)}$ $x - 7 = -1(7 -$

$= \dfrac{(x+2)}{-(x+4)}$ or $-\dfrac{x+2}{x+4}$ Simplify.

ANSWER:

$-\dfrac{x+2}{x+4}$

8. $\dfrac{9x^2-x^3}{x^2-3x-54}$

SOLUTION:

$\dfrac{9x^2-x^3}{x^2-3x-54} = \dfrac{x^2(9-x)}{x^2-3x-54}$ Distributive Proper

$= \dfrac{x^2(9-x)}{(x-9)(x+6)}$ Factor.

$= \dfrac{x^2(-1)(x-9)}{(x-9)(x+6)}$ $x - 9 = -1(9 - x)$

$= \dfrac{-x^2}{x+6}$ Simplify.

ANSWER:

$-\dfrac{x^2}{x+6}$

15	Simplify rational expressions	1-16	315	Solution

9. $\dfrac{(x-4)(x^2+2x-48)}{(36-x^2)(x^2+4x-32)}$

SOLUTION:

$$\frac{(x-4)(x^2+2x-48)}{(36-x^2)(x^2+4x-32)} = \frac{(x-4)(x+8)(x-6)}{(6-x)(6+x)(x+8)(x-4)}$$ Factor.

$$= \frac{(x-4)(x-6)(x+8)}{-1(x-6)(6+x)(x+8)(x-4)}$$ $x-6=-1(6-x)$

$$= \frac{1}{-1(6+x)} \text{ or } -\frac{1}{x+6}$$ Simplify.

ANSWER:

$-\dfrac{1}{x+6}$

10. $\dfrac{16-c^2}{c^2+c-20}$

SOLUTION:

$$\frac{16-c^2}{c^2+c-20} = \frac{(4-c)(4+c)}{(c-4)(c+5)}$$ Factor.

$$= \frac{(-1)(c-4)(4+c)}{(c-4)(c+5)}$$ $c-4=-1(4-c)$

$$= \frac{-(4+c)}{(c+5)} \text{ or } -\frac{c+4}{c+5}$$ Simplify.

ANSWER:

$-\dfrac{c+4}{c+5}$

11. $\dfrac{3ac^3f^3}{8a^2bcf^4} \cdot \dfrac{12ab^2c}{18ab^3c^2f}$

SOLUTION:

ANSWER:

$\dfrac{c}{4ab^2f^2}$

12. $\dfrac{14xy^2z^3}{21w^4x^2z} \cdot \dfrac{7wxyz}{12w^2y^3z}$

SOLUTION:

ANSWER:

$\dfrac{7z^2}{18w^5}$

13. $\dfrac{64a^2b^5}{35b^2c^3f^4} \div \dfrac{12a^4b^3c}{70abcf^2}$

SOLUTION:

ANSWER:

$\dfrac{32b}{3ac^3f^2}$

14. $\dfrac{9x^2yz}{5z^4} \div \dfrac{12x^4y^2}{50xy^4z^2}$

SOLUTION:

ANSWER:

$\dfrac{15y^3}{2xz}$

15. $\dfrac{15a^2b^2}{21ac} \cdot \dfrac{14a^4c^2}{6ab^3}$

SOLUTION:

ANSWER:

$\dfrac{5a^4c}{3b}$

16. $\dfrac{64a^2b^5}{35b^2c^3f^4} \div \dfrac{12a^4b^3c}{70abcf^2}$

SOLUTION:

ANSWER:

$\dfrac{32b}{3ac^3f^2}$

16	Simplify complex algebraic fractions including rational expressions	16-19	323

Examples 4 and 5

Simplify each expression.

16. $\dfrac{\frac{2}{x-3}+\frac{3x}{x^2-9}}{\frac{3}{x+3}-\frac{4x}{x^2-9}}$

17. $\dfrac{\frac{4}{x+5}+\frac{9}{x-6}}{\frac{5}{x-6}-\frac{8}{x+5}}$

18. $\dfrac{\frac{5}{x+6}-\frac{2x}{2x-1}}{\frac{x}{2x-1}+\frac{4}{x+6}}$

19. $\dfrac{\frac{8}{x-9}-\frac{x}{3x+2}}{\frac{3}{3x+2}+\frac{4x}{x-9}}$

16	Simplify complex algebraic fractions including rational expressions	16-19	323

solution method

Lesson 7-2

Adding and Subtracting Rational Expressions

Learn Adding and Subtracting Rational Expressions

Just as with rational numbers in fractional form, to add or subtract two rational expressions that have unlike denominators, you must first find the least common denominator (LCD). The LCD is the least common multiple (LCM) of the two denominators.

To find the LCM of two or more numbers or polynomials, factor them. The LCM contains each factor the greatest number of times it appears as a factor.

Key Concept • Adding Rational Expressions

Words: To add rational expressions, find the least common denominator. Rewrite each expression with the LCD. Then add.

Symbols: For all rational expressions $\frac{a}{b}$ and $\frac{c}{d}$ with $b \neq 0$ and $d \neq 0$, $\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}$.

Key Concept • Subtracting Rational Expressions

Words: To subtract rational expressions, find the least common denominator. Rewrite each expression with the LCD. Then subtract.

Symbols: For all rational expressions $\frac{a}{b}$ and $\frac{c}{d}$ with $b \neq 0$ and $d \neq 0$, $\frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{bc}{bd} = \frac{ad - bc}{bd}$.

Example 3 Use Addition and Subtraction of Rational Expressions

Learn Simplifying Complex Fractions

Complex fractions can be simplified by simplifying the numerator and denominator separately and then simplifying the resulting expression. You can also simplify a complex fraction by finding the LCD of all of the denominators. Then, the denominators can all be eliminated by multiplying by the LCD.

Example 4 Simplify Complex Fractions by Using Different LCDs

16	Simplify complex algebraic fractions including rational expressions	16-19	323

Solution

Simplify each expression.

16. $\dfrac{\dfrac{2}{x-3}+\dfrac{3x}{x^2-9}}{\dfrac{3}{x+3}-\dfrac{4x}{x^2-9}}$

SOLUTION:

$$\frac{\frac{2}{x-3}+\frac{3x}{x^2-9}}{\frac{3}{x+3}-\frac{4x}{x^2-9}}=\frac{\left(\frac{2}{x-3}+\frac{3x}{x^2-9}\right)}{\left(\frac{3}{x+3}-\frac{4x}{x^2-9}\right)}\cdot\frac{(x+3)(x-3)}{(x+3)(x-3)}$$ The LCD is x^2-9 or $(x+$

$$=\frac{2(x+3)+3x}{3(x-3)-4x}$$ Distributive Property

$$=\frac{2x+6+3x}{3x-9-4x}$$ Multiply.

$$=\frac{5x+6}{-x-9}$$ Simplify.

ANSWER:

$$\frac{5x+6}{-x-9}$$

17. $\dfrac{\dfrac{4}{x+5}+\dfrac{9}{x-6}}{\dfrac{5}{x-6}-\dfrac{8}{x+5}}$

SOLUTION:

$$\frac{\frac{4}{x+5}+\frac{9}{x-6}}{\frac{5}{x-6}-\frac{8}{x+5}}=\frac{\left(\frac{4}{x+5}+\frac{9}{x-6}\right)}{\left(\frac{5}{x-6}-\frac{8}{x+5}\right)}\cdot\frac{(x+5)(x-6)}{(x+5)(x-6)}$$ The LCD is $(x$

$$=\frac{4(x-6)+9(x+5)}{5(x+5)-8(x-6)}$$ Distributive Pr

$$=\frac{4x-24+9x+45}{5x+25-8x+48}$$ Multiply.

$$=\frac{13x+21}{-3x+73}$$ Simplify.

ANSWER:

$$\frac{13x+21}{-3x+73}$$

18. $\dfrac{\dfrac{5}{x+6}-\dfrac{2x}{2x-1}}{\dfrac{x}{2x-1}+\dfrac{4}{x+6}}$

SOLUTION:

$$\frac{\frac{5}{x+6}-\frac{2x}{2x-1}}{\frac{x}{2x-1}+\frac{4}{x+6}}=\frac{\left(\frac{5}{x+6}-\frac{2x}{2x-1}\right)}{\left(\frac{x}{2x-1}+\frac{4}{x+6}\right)}\cdot\frac{(2x-1)(x+6)}{(2x-1)(x+6)}$$ The LCD is

$$=\frac{5(2x-1)-2x(x+6)}{x(x+6)+4(2x-1)}$$ Distributive

$$=\frac{10x-5-2x^2-12x}{x^2+6x+8x-4}$$ Multiply.

$$=\frac{-2x^2-2x-5}{x^2+14x-4}$$ Simplify.

ANSWER:

$$\frac{-2x^2-2x-5}{x^2+14x-4}$$

19. $\dfrac{\dfrac{8}{x-9}-\dfrac{x}{3x+2}}{\dfrac{3}{3x+2}+\dfrac{4x}{x-9}}$

SOLUTION:

$$\frac{\frac{8}{x-9}-\frac{x}{3x+2}}{\frac{3}{3x+2}+\frac{4x}{x-9}}=\frac{\left(\frac{8}{x-9}-\frac{x}{3x+2}\right)}{\left(\frac{3}{3x+2}+\frac{4x}{x-9}\right)}\cdot\frac{(x-9)(3x+2)}{(x-9)(3x+2)}$$ The LCD

$$=\frac{8(3x+2)-x(x-9)}{3(x-9)+4x(3x+2)}$$ Distributi

$$=\frac{24x+16-x^2+9x}{3x-27+12x^2+8x}$$ Multiply.

$$=\frac{-x^2+33x+16}{12x^2+11x-27}$$ Simplify.

ANSWER:

$$\frac{-x^2+33x+16}{12x^2+11x-27}$$

17	Determine properties of reciprocal functions	51-56	336

51. **CREATE** Write a reciprocal function for which the graph has a vertical asymptote at $x = -4$ and a horizontal asymptote at $f(x) = 6$.

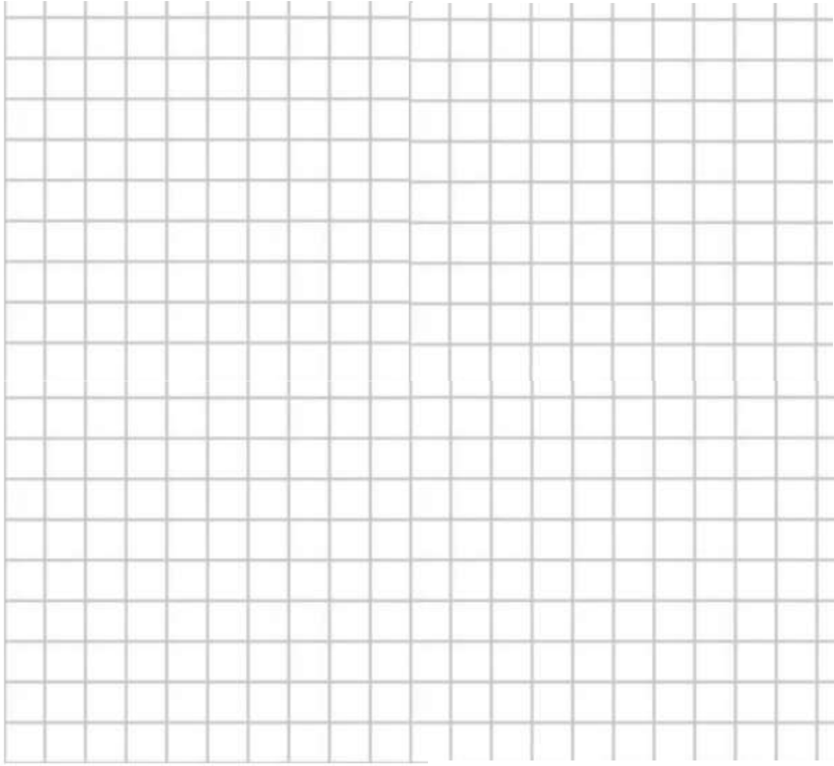
17	Determine properties of reciprocal functions	51-56	336

52. **ANALYZE** Consider the functions $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$.

a. Make a table of values comparing the two functions. Then graph both functions.

b. Compare and contrast the two graphs.

c. Make a conjecture about the difference between the graphs of reciprocal functions with an even exponent in the denominator and those with an odd exponent in the denominator.

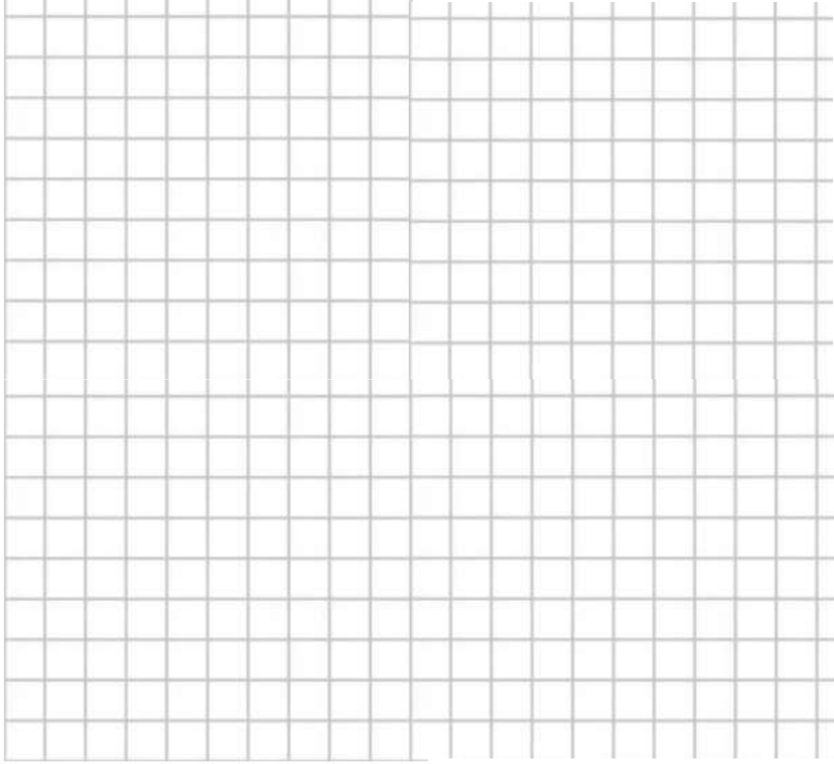


17	Determine properties of reciprocal functions	51-56	336

53. **WHICH ONE DOESN'T BELONG?** Find the function that does not belong. Justify your conclusion.

$f(x) = \frac{3}{x+1}$ $g(x) = \frac{x+2}{x^2+1}$ $h(x) = \frac{5}{x^2+2x+1}$ $j(x) = \frac{20}{x-7}$

54. **PERSEVERE** Write two different reciprocal functions with graphs having the same vertical and horizontal asymptotes. Then graph the functions.



17	Determine properties of reciprocal functions	51-56	336

55. PERSEVERE Graph $f(x) = \frac{4}{(x + 2)^2}$. What are the asymptotes of the graph?

56. WRITE Explain why only part of the graph of a rational function may be meaningful in a real-world situation in the context of the problem.

17	Determine properties of reciprocal functions	51-56	336

Solution

solution method

Lesson 7-3

Graphing Reciprocal Functions

Learn Graphing Reciprocal Functions

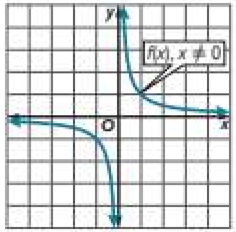
A **reciprocal function** has an equation of the form $f(x) = \frac{n}{b(x)}$, where n is a real number and $b(x)$ is a linear expression that cannot equal 0.

The parent function of a reciprocal function is $f(x) = \frac{1}{x}$. A **vertical asymptote** is a vertical line that a graph approaches. A **horizontal asymptote** is a horizontal line that a graph approaches. Because the function $f(x) = \frac{1}{x}$ is not defined when $x = 0$, there is a vertical asymptote at $x = 0$. The type of graph formed by a reciprocal function is called a **hyperbola**.

The domain of a function is limited to values for which the function is defined. Values for which the function is not defined are called **excluded values**.

Key Concept • Reciprocal Functions

Parent function	$f(x) = \frac{1}{x}$
Type of graph	hyperbola
Domain and range	all nonzero real numbers
Asymptotes	$x = 0$ and $f(x) = 0$
Intercepts	none
Not defined	$x = 0$



A reciprocal function has two asymptotes, which are lines that a graph approaches. The vertical asymptote is determined by the excluded value of x, and the horizontal asymptote is determined by the value that is undefined for $f(x)$.

For a reciprocal function in the form $f(x) = \frac{n}{b(x)}$, the horizontal asymptote is $f(x) = 0$ because there is no value of x that will result in $f(x) = 0$. For a reciprocal function of the form $f(x) = \frac{n}{b(x)} + k$, where k is a constant, the horizontal asymptote is $f(x) = k$.

Example 3 Analyze a Reciprocal Function

51. **CREATE** Write a reciprocal function for which the graph has a vertical asymptote at $x = -4$ and a horizontal asymptote at $f(x) = 6$.

SOLUTION:

Using the function $f(x) = \frac{a}{x-h} + k$, substitute -4 for h and 6 for k to get $f(x) = \frac{1}{x+4} + 6$.

ANSWER:

Sample answer: $f(x) = \frac{1}{x+4} + 6$

For Exercises 44 and 45, refer to the equation $y = -\frac{4}{5}x + \frac{2}{5}$ where $-2 \le x \le 5$.

52. **ANALYZE** Consider the functions $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$.

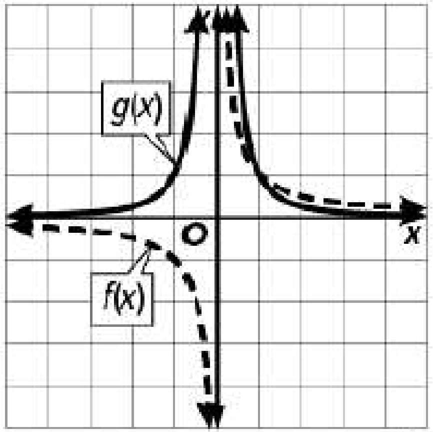
a. Make a table of values comparing the two functio
Then graph both functions.
b. Compare and contrast the two graphs.
c. Make a conjecture about the difference between t
graphs of reciprocal functions with an even exponent
the denominator and those with an odd exponent in t
denominator.

SOLUTION:

SOLUTION:

a.

$f(x) = 1/x$		$g(x) = 1/x^2$	
x	$f(x)$	x	$g(x)$
–3	$-\frac{1}{3}$	–3	$\frac{1}{9}$
–2	$-\frac{1}{2}$	–2	$\frac{1}{4}$
–1	–1	–1	1
0	undefined	0	undefine
1	1	1	1
2	$\frac{1}{2}$	2	$\frac{1}{4}$
3	$\frac{1}{3}$	3	$\frac{1}{9}$



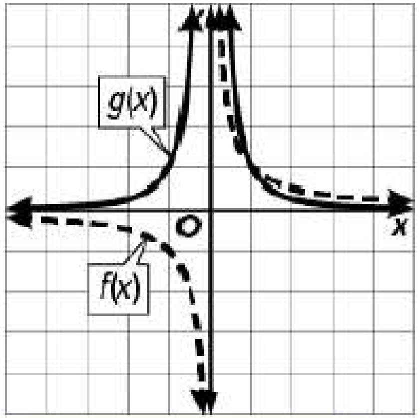
17	Determine properties of reciprocal functions	51-56	336	Solution

b. The positive portion of $f(x) = \frac{1}{x^2}$ is similar to the graph of $f(x) = \frac{1}{x}$. Positive values of x produce pos values of $f(x)$. The negative portion of $f(x) = \frac{1}{x^2}$ appears to be a reflection of $f(x) = \frac{1}{x}$ over the x-a Negative values of x produce positive values of $f(x)$.
c. Sample answer: When the exponent is even, the g will show a reflection over the x-axis. When the exponent is odd, the graph will show a reflection over $y = x$.

ANSWER:

a.

$f(x) = 1/x$		$g(x)=1/x^2$	
x	$f(x)$	x	$g(x)$
–3	$-\frac{1}{3}$	–3	$\frac{1}{9}$
–2	$-\frac{1}{2}$	–2	$\frac{1}{4}$
–1	–1	–1	1
0	undefined	0	undefin
1	1	1	1
2	$\frac{1}{2}$	2	$\frac{1}{4}$
3	$\frac{1}{3}$	3	$\frac{1}{9}$



b. The positive portion of $f(x) = \frac{1}{x^2}$ is similar to the graph of $f(x) = \frac{1}{x}$. Positive values of x produce pos values of $f(x)$. The negative portion of $f(x) = \frac{1}{x^2}$ appears to be a reflection of $f(x) = \frac{1}{x}$ over the x-a Negative values of x produce positive values of $f(x)$.
c. Sample answer: When the exponent is even, the g will show a reflection over the x-axis. When the exponent is odd, the graph will show a reflection ove $= x$.

53. **WHICH ONE DOESN'T BELONG** Find the fu not belong with the other three. Justify your conclusi

$f(x) = \frac{3}{x+1}$	$g(x) = \frac{x+2}{x^2+1}$	$h(x) = \frac{5}{x^2+2x+1}$

SOLUTION:

$g(x)$ is the function that doesn't belong because it has in both its numerator and denominator; whereas all o unknowns only in the denominator.

ANSWER:

Sample answer: $g(x)$; All other choices have unknow denominator.

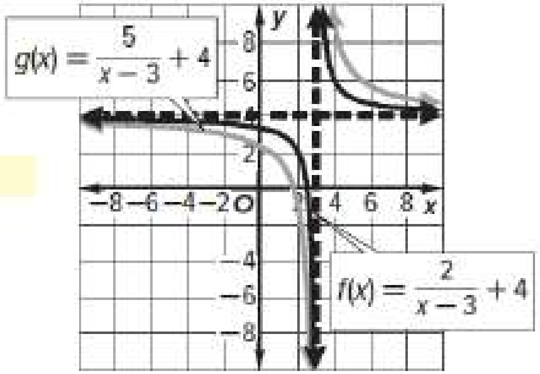
17	Determine properties of reciprocal functions	51-56	336

Solution

54. **PERSEVERE** Write two different reciprocal functions with graphs having the same vertical and horizontal asymptotes. Then graph the functions.

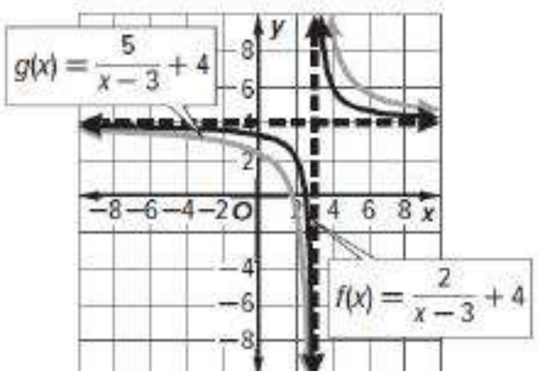
SOLUTION:

$f(x) = \frac{2}{x-3} + 4$ and $g(x) = \frac{5}{x-3} + 4$; both have a value of $h = 3$ and a value of $k = 4$.



ANSWER:

Sample answer: $f(x) = \frac{2}{x-3} + 4$ and $g(x) = \frac{5}{x-3} + 4$;



18	Graph rational functions with oblique asymptotes and point discontinuity	11-30	344

Example 4

Find the zeros and asymptotes of each function. Then graph each function.

11. $f(x) = \dfrac{(x-4)^2}{x+2}$

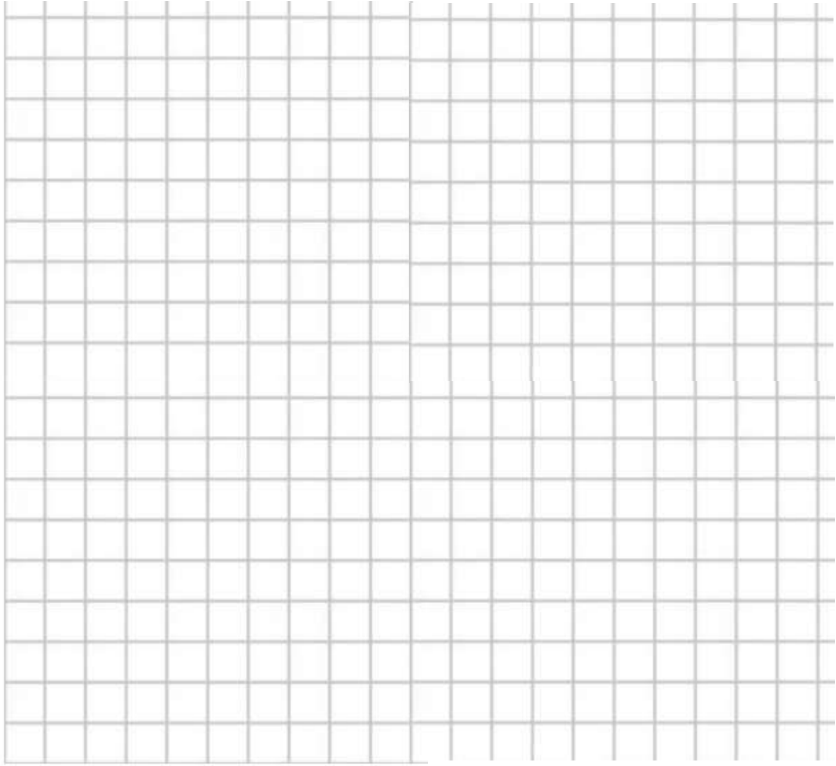
12. $f(x) = \dfrac{(x+3)^2}{x-5}$

18	Graph rational functions with oblique asymptotes and point discontinuity	11-30	344

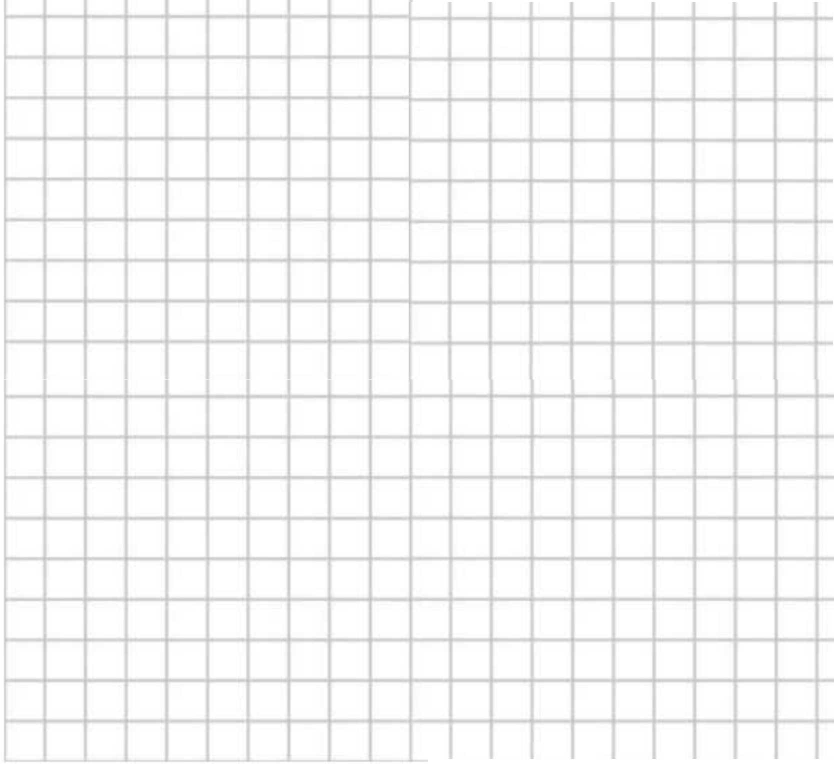
Example 4

Find the zeros and asymptotes of each function. Then graph each function.

13. $f(x) = \dfrac{6x^2 + 4x + 2}{x + 2}$



14. $f(x) = \dfrac{2x^2 + 7x}{x - 2}$



18	Graph rational functions with oblique asymptotes and point discontinuity	11-30	344

Example 4

Find the zeros and asymptotes of each function. Then graph each function.

15. $f(x) = \dfrac{3x^2 + 8}{2x - 1}$

16. $f(x) = \dfrac{2x^2 + 5}{3x + 4}$

18	Graph rational functions with oblique asymptotes and point discontinuity	11-30	344

Example 5

Graph each function. Find the point discontinuity.

17. $f(x) = \frac{x^2 - 2x - 8}{x - 4}$

18. $f(x) = \frac{x^2 + 4x - 12}{x - 2}$

18	Graph rational functions with oblique asymptotes and point discontinuity	11-30	344

Example 5

Graph each function. Find the point discontinuity.

19. $f(x) = \frac{x^2 - 25}{x + 5}$

20. $f(x) = \frac{x^2 - 64}{x - 8}$

18	Graph rational functions with oblique asymptotes and point discontinuity	11-30	344

Example 5

Graph each function. Find the point discontinuity.

21. $f(x) = \dfrac{(x-4)(x^2-4)}{x^2-6x+8}$

22. $f(x) = \dfrac{(x+5)(x^2+2x-3)}{x^2+8x+15}$

18	Graph rational functions with oblique asymptotes and point discontinuity	11-30	344

Mixed Exercises

Graph each function.

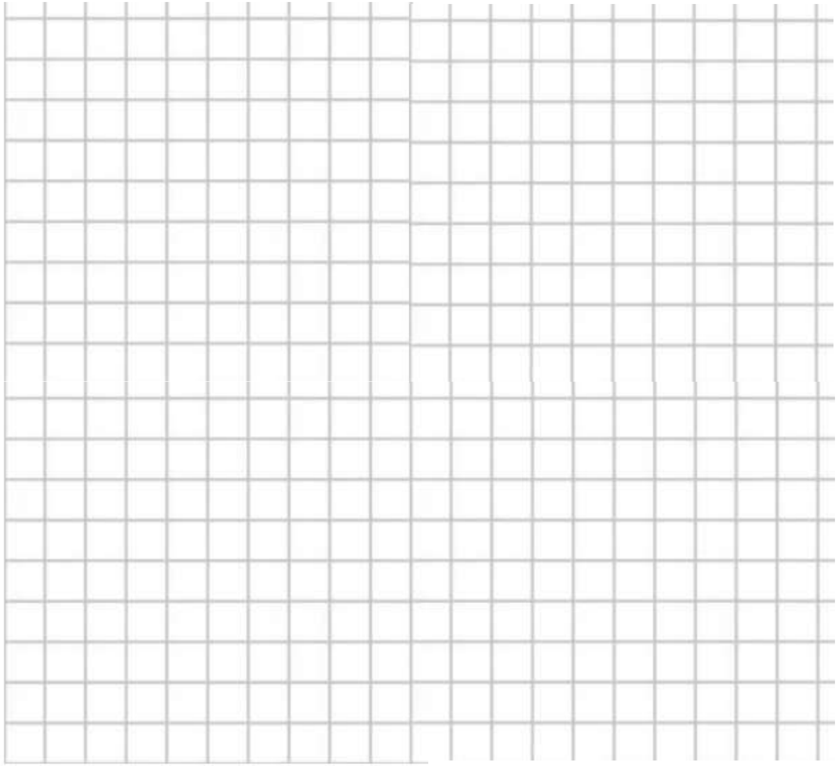
23. $f(x) = \frac{x}{x + 2}$

24. $f(x) = \frac{x^2 - 4}{x - 2}$

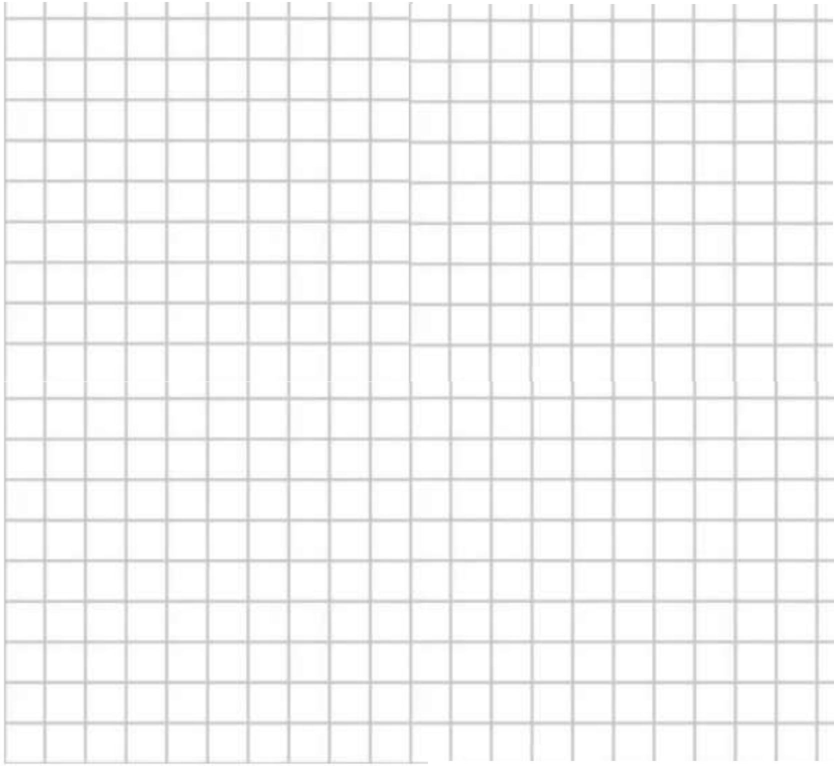
18	Graph rational functions with oblique asymptotes and point discontinuity	11-30	344

Graph each function.

25. $f(x) = \dfrac{x^2 + x - 12}{x - 3}$



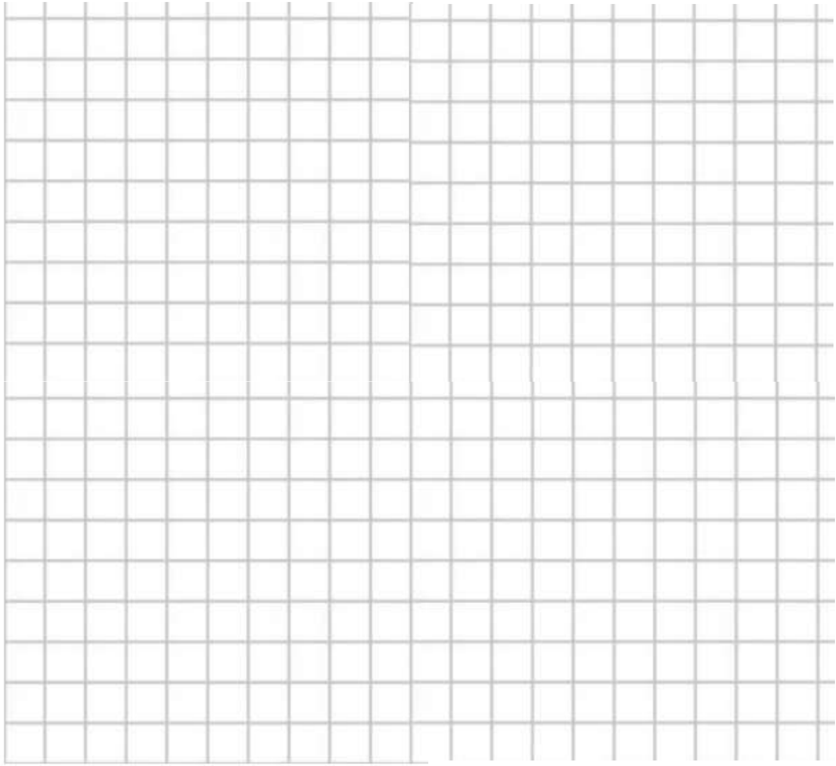
26. $f(x) = \dfrac{x - 1}{x^2 - 4x + 3}$



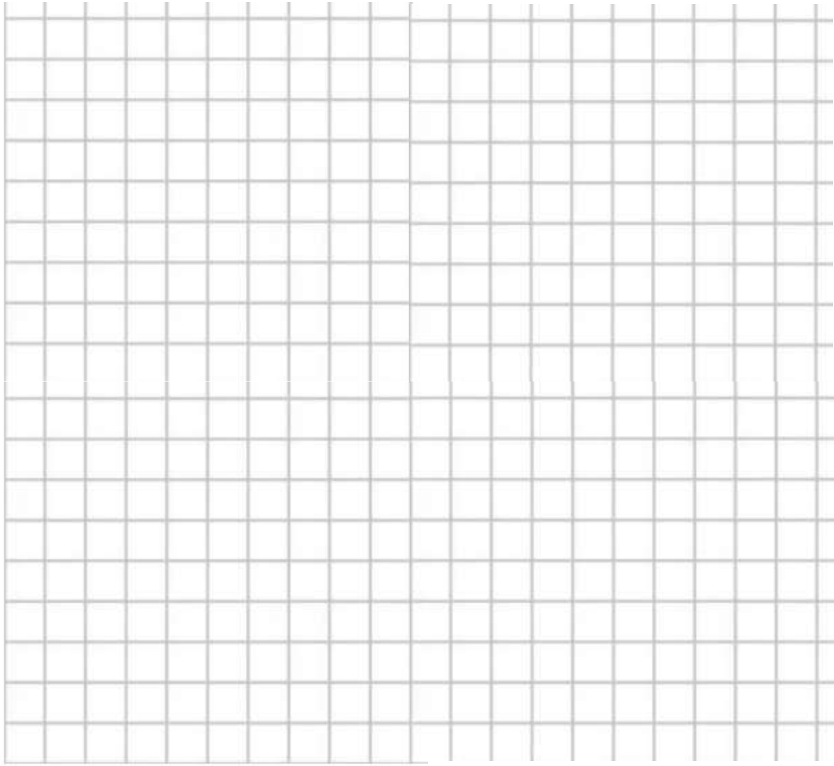
18	Graph rational functions with oblique asymptotes and point discontinuity	11-30	344

Graph each function.

27. $f(x) = \dfrac{3}{x^2 - 2x - 8}$



28. $f(x) = \dfrac{x^3}{2x + 2}$

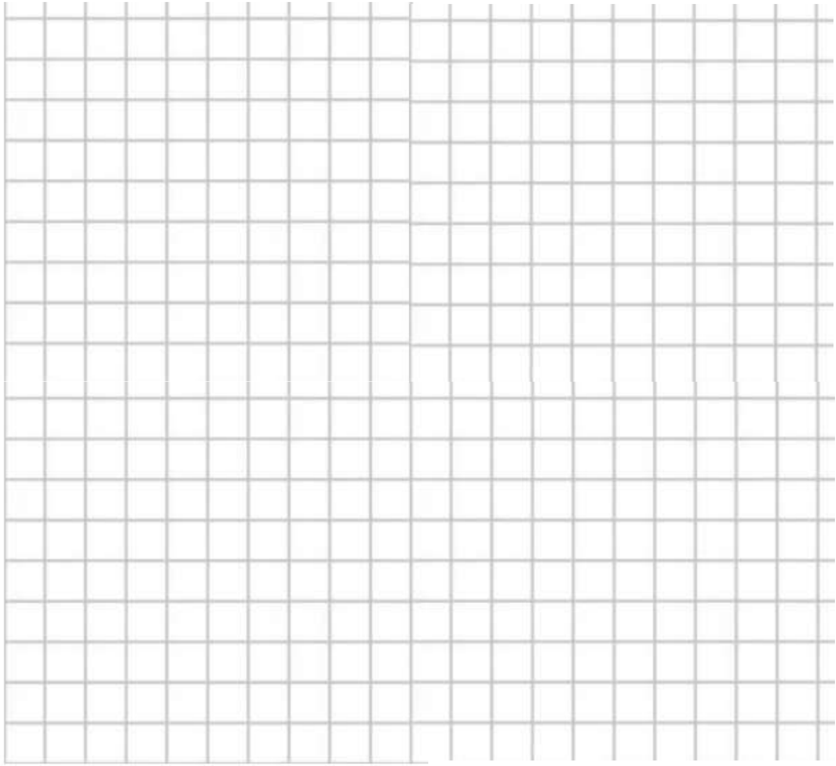


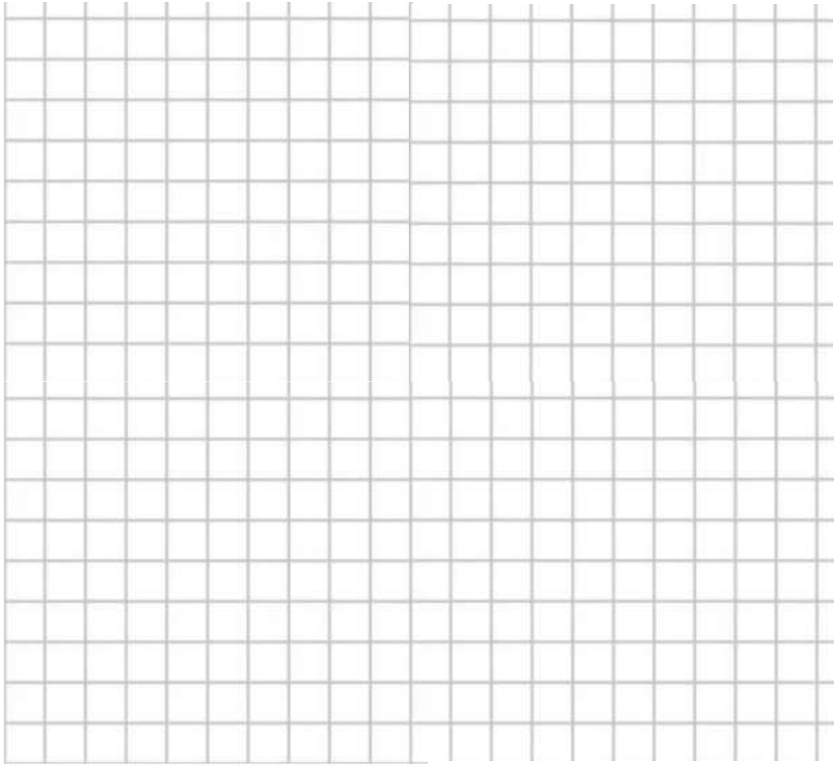
18	Graph rational functions with oblique asymptotes and point discontinuity	11-30	344

Graph each function.

29. $f(x) = \dfrac{2x^3 + 4x^2 - 10x - 12}{2x^2 + 8x + 6}$

30. $f(x) = \dfrac{(x + 1)^2}{2x - 1}$





18	Graph rational functions with oblique asymptotes and point discontinuity	11-30	344

Solution

solution method

Lesson 7-4

Graphing Rational Functions

Learn Graphing Rational Functions with Oblique Asymptotes

An **oblique asymptote**, or slant asymptote, is neither horizontal nor vertical.

Key Concept • Oblique Asymptotes

If $f(x) = \frac{a(x)}{b(x)}$, where $a(x)$ and $b(x)$ are polynomial functions with no common factors other than 1 and $b(x) \neq 0$, then $f(x)$ has an oblique asymptote if the degree of $a(x)$ minus the degree of $b(x)$ equals 1. The equation of the asymptote is $f(x) = \frac{a(x)}{b(x)}$ with no remainder.

In some cases, graphs of rational functions may have **point discontinuity**, which looks like a hole in the graph. This is because the function is undefined at that point. If the original function is undefined for $x = a$ but the related rational expression of the function in simplest form is defined for $x = a$, then there is a point discontinuity or hole in the graph at $x = a$.

Key Concept • Point Discontinuity

If $f(x) = \frac{a(x)}{b(x)}$, $b(x) \neq 0$, and $x - c$ is a factor of both $a(x)$ and $b(x)$, then there is a point discontinuity at $x = c$.

Example 4 Graph with Oblique Asymptotes

Example 5 Graph with Point Discontinuity

Find the zeros and asymptotes of each function. Then graph each function.

11. $f(x) = \frac{(x-4)^2}{x+2}$

SOLUTION:

Step 1 Find the zeros.

Set $a(x) = 0$

$$(x-4)^2 = 0$$
$$x - 4 = 0$$
$$x = 4$$

There is a zero at $x = 4$.

Step 2 Find the asymptotes.

Find the vertical asymptote. Set $b(x) = 0$.

$$x + 2 = 0$$
$$x = -2$$

Because the degree of the numerator is greater than the degree of the denominator, there is no horizontal asymptote.

The difference between the degree of the numerator and the degree of the denominator is 1, so there is an oblique asymptote. To find the oblique asymptote, divide the numerator by the denominator.

$$\begin{array}{r} x - 10 \\ x+2\overline{)x^2 - 8x + 16} \\ \underline{(-)\,x^2 + 2x} \\ -10x + 16 \\ \underline{(-)10x - 20} \\ 36 \end{array}$$

excluding the remainder. So, there is an oblique asymptote at $f(x) = x - 10$.

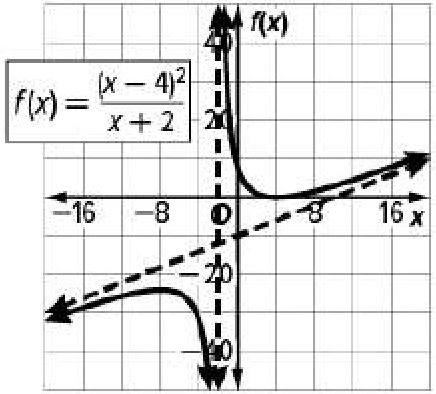
Step 3 Draw the graph.

Graph the asymptotes. Then make a table of values, and graph.

x	$f(x)$
–5	–27
–4	–32
–3	–49
–1	25
0	8
1	3
2	1
3	0.2
4	0

18	Graph rational functions with oblique asymptotes and point discontinuity	11-30	344

Solution



ANSWER:

zero: $x = 4$; vertical asymptote: $x = -2$; oblique asymptote: $f(x) = x - 10$

12. $f(x) = \frac{(x+3)^2}{x-5}$

SOLUTION:

Step 1 Find the zeros.

Set $a(x) = 0$

$(x+3)^2 = 0$

$x + 3 = 0$

$x = -3$

There is a zero at $x = -3$.

Step 2 Find the asymptotes.

Find the vertical asymptote. Set $b(x) = 0$.

$x - 5 = 0$

$x = 5$

Because the degree of the numerator is greater than the degree of the denominator, there is no horizontal asymptote.

The difference between the degree of the numerator and the degree of the denominator is 1, so there is an oblique asymptote. To find the oblique asymptote, divide the numerator by the denominator.

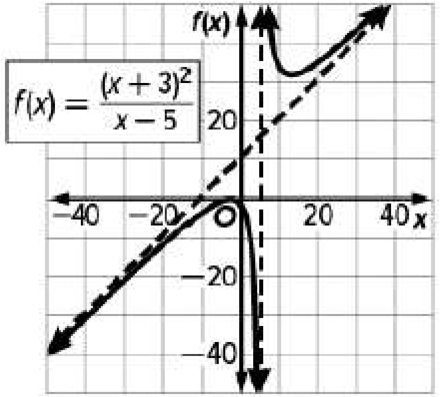
$$\begin{array}{r} x + 11 \\ x-5\overline{)x^2+6x+9} \\ (-)\,x^2-5x \\ 11x+9 \\ (-)11x-55 \\ 64 \end{array}$$

The equation of the asymptote is the quotient excluding the remainder. So, there is an oblique asymptote at $f(x) = x + 11$.

Step 3 Draw the graph.

Graph the asymptotes. Then make a table of values, and graph.

x	$f(x)$
–40	–30.42
–27	–18
–15	–7.2
–11	–4
1	–4
9	36
21	36
25	39.2
37	50



ANSWER:

zero: $x = -3$; vertical asymptote: $x = 5$; oblique asymptote: $f(x) = x + 11$

18	Graph rational functions with oblique asymptotes and point discontinuity	11-30	344

Solution

13. $f(x) = \dfrac{6x^2+4x+2}{x+2}$

SOLUTION:

Step 1 Find the zeros.

$$\text{Set } a(x) = 0$$
$$6x^2+4x+2=0$$
$$2(3x^2+2x+1)=0$$
$$2(3x-1)(x+1)=0$$
$$x+1=0 \text{ or } 3x-1=0 \quad \text{Zero Product Property}$$
$$x=-1 \text{ or } x=\tfrac{1}{3} \quad \text{Solve each equation.}$$

Step 2 Find the asymptotes.

Find the vertical asymptote. Set $b(x) = 0$.

$$x+2=0$$
$$x=-2$$

Because the degree of the numerator is greater than the degree of the denominator, there is no horizontal asymptote.

The difference between the degree of the numerator and the degree of the denominator is 1, so there is an oblique asymptote. To find the oblique asymptote, divide the numerator by the denominator.

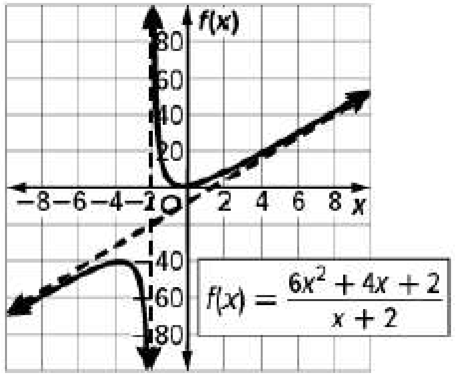
$$\begin{array}{r} 6x-8 \\ x+2\overline{)6x^2+4x+2} \\ \underline{(-)\,6x^2+12x} \\ -8x+2 \\ \underline{(-)-8x-16} \\ -14 \end{array}$$

The equation of the asymptote is the quotient excluding the remainder. So, there is an oblique asymptote at $f(x) = 6x - 8$.

Step 3 Draw the graph.

Graph the asymptotes. Then make a table of values, and graph.

x	$f(x)$
–6	–48.5
–5	–44
–4	–41
–3	–44
–1	4
0	1
1	4
2	8.5
4	19



ANSWER:

zeros: $x = -1$ or $x = \dfrac{1}{3}$; vertical asymptote: $x = -2$; oblique asymptote: $f(x) = 6x - 8$

14. $f(x) = \dfrac{2x^2+7x}{x-2}$

SOLUTION:

Step 1 Find the zeros.

$$\text{Set } a(x) = 0$$
$$2x^2+7x=0$$
$$x(2x+7)=0$$
$$x=0 \text{ or } 2x+7=0 \quad \text{Zero Product Property}$$
$$x=0 \text{ or } x=-3.5 \quad \text{Solve each equation.}$$

There is a zero at $x = 0$. There is a zero at $x = -3.5$.

Step 2 Find the asymptotes.

Find the vertical asymptote. Set $b(x) = 0$.

$$x-2=0$$
$$x=2$$

Because the degree of the numerator is greater than the degree of the denominator, there is no horizontal asymptote.

The difference between the degree of the numerator and the degree of the denominator is 1, so there is an oblique asymptote. To find the oblique asymptote, divide the numerator by the denominator.

$$\begin{array}{r} 2x+11 \\ x-2\overline{)x^2+7x+0} \\ \underline{(-)\,x^2-4x} \\ 11x+0 \\ \underline{(-)11x-22} \\ 22 \end{array}$$

The equation of the asymptote is the quotient excluding the remainder. So, there is an oblique asymptote at $f(x) = 2x + 11$.

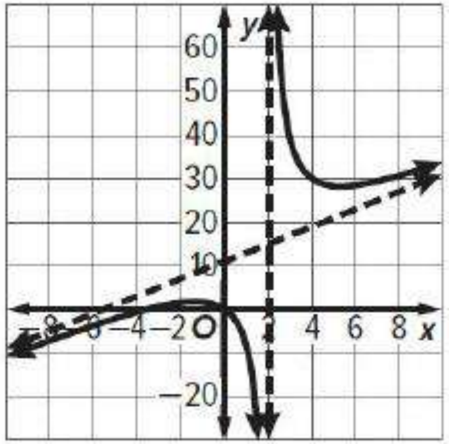
Step 3 Draw the graph.

Graph the asymptotes. Then make a table of values, and graph.

18	Graph rational functions with oblique asymptotes and point discontinuity	11-30	344

Solution

x	$f(x)$
–4	≈ –0.67
–3	0.6
–2	1.5
–1	≈ 1.67
0	0
1	–9
3	39



ANSWER:

zeros: $x = 0$ and $x = -3.5$; vertical asymptote: $x = 2$; oblique asymptote: $f(x) = 2x + 11$

15. $f(x) = \frac{3x^2+8}{2x-1}$

SOLUTION:

Step 1 Find the zeros.

Set $a(x) = 0$

$3x^2 + 8 = 0$

$3x^2 = -8$

$x^2 = -\frac{8}{3}$

There is no zero since there is no real number that is the square root of a negative number.

Step 2 Find the asymptotes.

Find the vertical asymptote. Set $b(x) = 0$.

$2x - 1 = 0$

$2x = 1$

$x = \frac{1}{2}$

Because the degree of the numerator is greater than the degree of the denominator, there is no horizontal asymptote.

The difference between the degree of the numerator and the degree of the denominator is 1, so there is an oblique asymptote. To find the oblique asymptote, divide the numerator by the denominator.

$$\frac{3}{2}x + \frac{3}{4}$$

$$2x-1\overline{)3x^2+0x+8}$$

$$(-)\ 3x^2 - \frac{3}{2}x$$

$$\frac{3}{2}x + 8$$

$$(-)\frac{3}{2}x - \frac{3}{4}$$

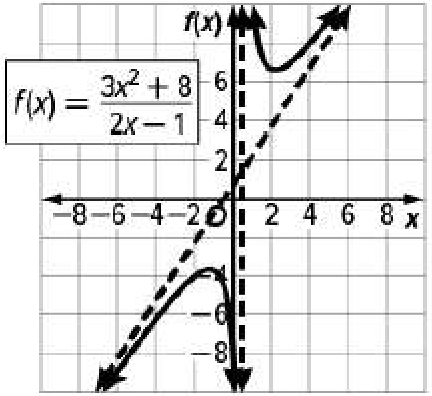
$$8\frac{3}{4}$$

The equation of the asymptote is the quotient excluding the remainder. So, there is an oblique asymptote at $f(x) = \frac{3}{2}x + \frac{3}{4}$.

Step 3 Draw the graph.

Graph the asymptotes. Then make a table of values, and graph.

x	$f(x)$
–6	≈ –8.9
–4	≈ –6.2
–3	–5
–2	–4
0	–8
1	11
3	7
4	8



ANSWER:

zero: none; vertical asymptote: $x = \frac{1}{2}$; oblique asymptote: $f(x) = \frac{3}{2}x + \frac{3}{4}$

16. $f(x) = \frac{2x^2+5}{3x+4}$

SOLUTION:

Step 1 Find the zeros.

Set $a(x) = 0$

$2x^2 + 5 = 0$

$2x^2 = -5$

$x^2 = -\frac{5}{2}$

There is no zero since there is no real number that is the square root of a negative number.

Step 2 Find the asymptotes.

Find the vertical asymptote. Set $b(x) = 0$.

$3x + 4 = 0$

$3x = -4$

$x = -\frac{4}{3}$

Because the degree of the numerator is greater than the degree of the denominator, there is no horizontal asymptote.

The difference between the degree of the numerator and the degree of the denominator is 1, so there is an oblique asymptote. To find the oblique asymptote, divide the numerator by the denominator.

$$\frac{2}{3}x - \frac{8}{9}$$

$$3x+4\overline{)2x^2+0x+5}$$

$$\underline{(-)\ 2x^2 + \frac{8}{3}x}$$

$$-\frac{8}{3}x + 5$$

$$\underline{(-) -\frac{8}{9}x - \frac{32}{49}}$$

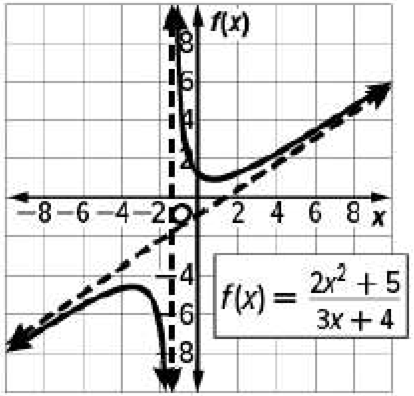
$$-\frac{32}{49}$$

The equation of the asymptote is the quotient excluding the remainder. So, there is an oblique asymptote at $f(x) = \frac{2}{3}x - \frac{8}{9}$.

Step 3 Draw the graph.

Graph the asymptotes. Then make a table of values, and graph.

x	$f(x)$
–6	–5.5
–5	–5
–3	–4.6
–2	–6.5
–1	7
0	1.25
1	1
3	$\approx 1\frac{3}{4}$
6	3.5



ANSWER:

zero: none; vertical asymptote: $x = -\frac{4}{3}$; oblique asymptote: $f(x) = \frac{2}{3}x - \frac{8}{9}$

Graph each function. Find the point discontinuity.

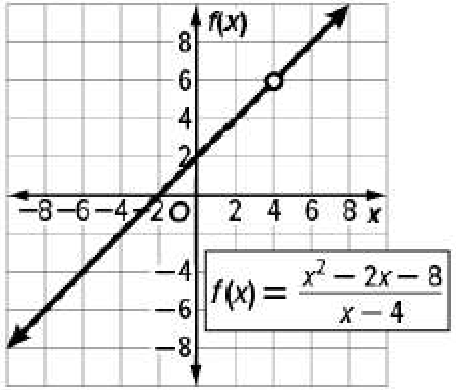
17. $f(x) = \frac{x^2-2x-8}{x-4}$

SOLUTION:

Notice that $\frac{x^2-2x-8}{x-4} = \frac{(x-4)(x+2)}{x-4}$ or $x + 2$.

However, because the denominator of the original function cannot be 0, there is a discontinuity at $x - 4 = 0$ or $x = 4$.

Therefore, the graph of $f(x) = \frac{x^2-2x-8}{x-4}$ is the graph of $f(x) = x + 2$ with a hole or point of discontinuity at $x = 4$.



ANSWER:

point discontinuity at $x = 4$

18	Graph rational functions with oblique asymptotes and point discontinuity	11-30	344

Solution

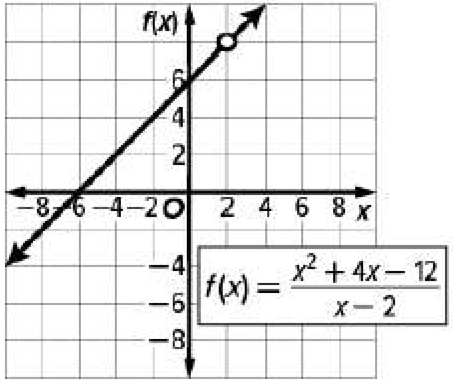
18. $f(x)=\frac{x^2+4x-12}{x-2}$

SOLUTION:

Notice that $\frac{x^2+4x-12}{x-2}=\frac{(x+6)(x-2)}{x-2}$ or $x+6$.

However, because the denominator of the original function cannot be 0, there is a discontinuity at $x + 6 = 0$ or $x = -6$.

Therefore, the graph of $f(x)=\frac{x^2+4x-12}{x-2}$ is the graph of $f(x) = x + 6$ with a hole or point of discontinuity at $x = -6$.



ANSWER:

point discontinuity at $x = 2$

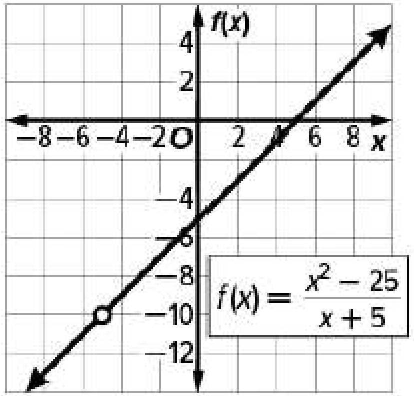
19. $f(x)=\frac{x^2-25}{x+5}$

SOLUTION:

Notice that $\frac{x^2-25}{x+5}=\frac{(x+5)(x-5)}{x+5}$ or $x-5$.

However, because the denominator of the original function cannot be 0, there is a discontinuity at $x + 5 = 0$ or $x = -5$.

Therefore, the graph of $f(x)=\frac{x^2-25}{x+5}$ is the graph of $f(x) = x + 5$ with a hole or point of discontinuity at $x = -5$.



ANSWER:

point discontinuity at $x = -5$

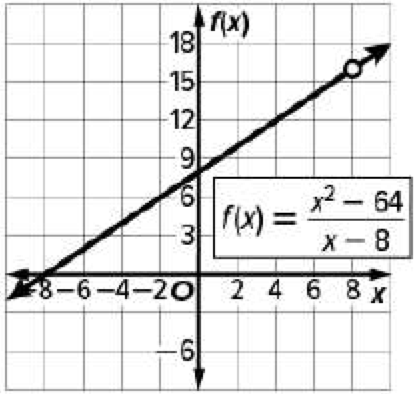
20. $f(x)=\frac{x^2-64}{x-8}$

SOLUTION:

Notice that $\frac{x^2-64}{x-8}=\frac{(x+8)(x-8)}{x-8}$ or $x+8$.

However, because the denominator of the original function cannot be 0, there is a discontinuity at $x - 8 = 0$ or $x = 8$.

Therefore, the graph of $f(x)=\frac{x^2-64}{x-8}$ is the graph of $f(x) = x - 8$ with a hole or point of discontinuity at $x = 8$.



ANSWER:

point discontinuity at $x = 8$

21. $f(x)=\frac{(x-4)(x^2-4)}{x^2-6x+8}$

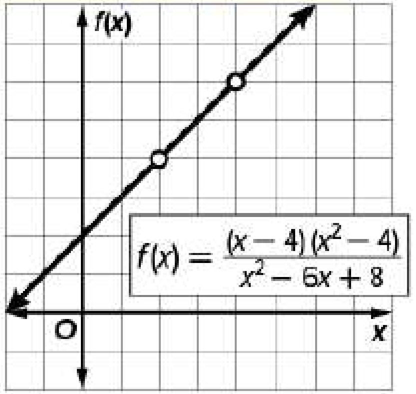
SOLUTION:

Notice that

$\frac{(x-4)(x^2-4)}{x^2-6x+8}=\frac{(x-4)(x+2)(x-2)}{(x-4)(x-2)}$ or $x+2$.

However, because the denominator of the original function cannot be 0, there is a discontinuity at $x - 4 = 0$ or $x = 4$ and $x - 2 = 0$ or $x = 2$.

Therefore, the graph of $f(x)=\frac{(x-4)(x^2-4)}{x^2-6x+8}$ is the graph of $f(x) = x + 2$ with a hole or point of discontinuity at $x = 4$ and $x = 2$.



ANSWER:

point discontinuity at $x = 2$ and $x = 4$

18	Graph rational functions with oblique asymptotes and point discontinuity	11-30	344

Solution

22. $f(x)=\frac{(x+5)(x^2+2x-3)}{x^2+8x+15}$

SOLUTION:

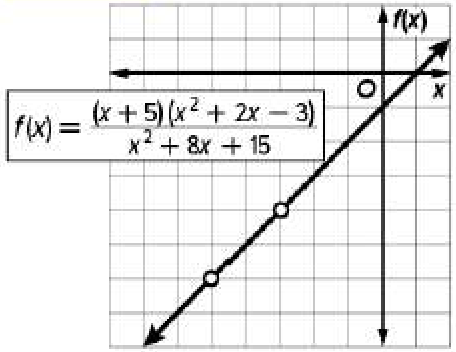
Notice that
$\frac{(x+5)(x^2+2x-3)}{x^2+8x+15}=\frac{(x+5)(x+3)(x-1)}{(x+5)(x+3)}$ or $x-1$.

However, because the denominator of the original function cannot be 0, there is a discontinuity at $x+5=0$ or $x=-5$ and $x+3=0$ or $x=-3$.

Therefore, the graph of

$f(x)=\frac{(x+5)(x^2+2x-3)}{x^2+8x+15}$ is the graph of $f(x)$

$=x-1$ with a hole or point of discontinuity at $x=-5$ and $x=-3$.



ANSWER:

point discontinuity at $x=-5$ and $x=-3$

23. $f(x)=\frac{x}{x+2}$

SOLUTION:

Step 1 Find the zeros.

There is a zero at $x=0$.

Step 2 Find the asymptotes.

Find the vertical asymptote. Set $b(x)=0$.

$x+2=0$

$x=-2$

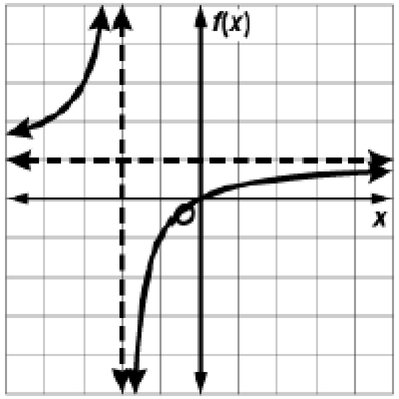
Because the degree of the numerator equals the degree of the denominator, the horizontal asymptote is the line

$y=\frac{\text{leading coefficient of } a(x)}{\text{leading coefficient of } b(x)}$, so $y=1$.

Step 3 Draw the graph.

Graph the asymptote. Then make a table of values, and graph.

x	$f(x)$
−5	$1\frac{2}{3}$
−4	2
−3	3
−1	−1
0	0
1	$\frac{1}{3}$
2	$\frac{1}{2}$
3	$\frac{3}{5}$
4	$\frac{2}{3}$



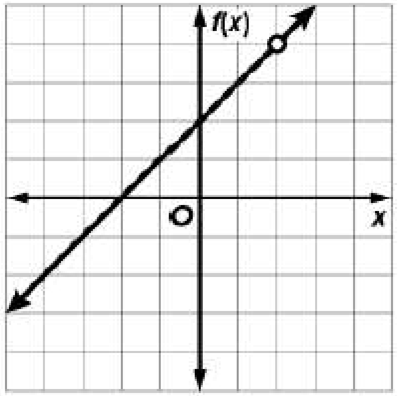
24. $f(x)=\frac{x^2-4}{x-2}$

SOLUTION:

Notice that $\frac{x^2-4}{x-2}=\frac{(x+2)(x-2)}{x-2}$ or $x+2$.

However, because the denominator of the original function cannot be 0, there is a discontinuity at $x-2=0$ or $x=2$.

Therefore, the graph of $f(x)=\frac{x^2-4}{x-2}$ is the graph of $f(x)=x-2$ with a hole or point of discontinuity at $x=2$.



18	Graph rational functions with oblique asymptotes and point discontinuity	11-30	344

Solution

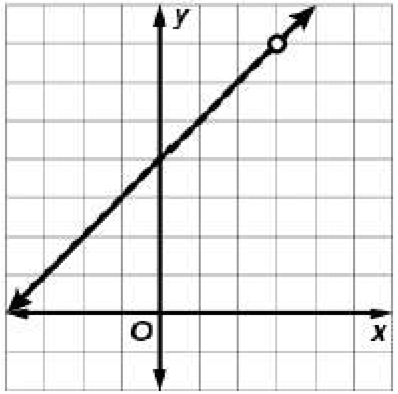
25. $f(x) = \frac{x^2+x-12}{x-3}$

SOLUTION:

Notice that $\frac{x^2+x-12}{x-3} = \frac{(x+4)(x-3)}{x-3}$ or $x+4$.

However, because the denominator of the original function cannot be 0, there is a discontinuity at $x - 3 = 0$ or $x = 3$.

Therefore, the graph of $f(x) = \frac{x^2+x-12}{x-3}$ is the graph of $f(x) = x - 3$ with a hole or point of discontinuity at $x = 3$.



26. $f(x) = \frac{x-1}{x^2-4x+3}$

SOLUTION:

Step 1 Find the zeros.

Set $a(x) = 0$

$x - 1 = 0$

$x = 1$

There is a zero at $x = 1$.

Step 2 Find the asymptotes.

Find the vertical asymptote. Set $b(x) = 0$.

$x^2 - 4x + 3 = 0$

$(x-1)(x-3) = 0$

$x - 1 = 0$ or $x - 3 = 0$ — Zero Product Property

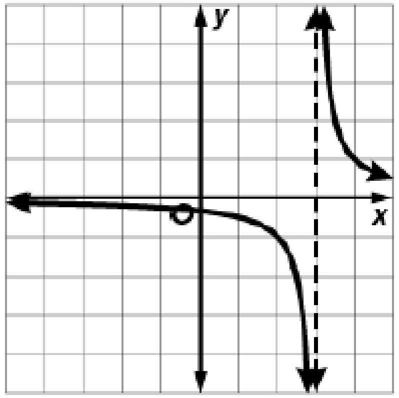
$x = 1$ or $x = 3$ — Solve each equation.

Because the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is the line $y = 0$.

Step 3 Draw the graph.

Graph the asymptote. Then make a table of values, and graph.

x	f(x)
–3	$-\frac{1}{6}$
–2	–0.2
–1	–0.25
0	$-\frac{1}{3}$
2	–1
4	1



27. $f(x) = \frac{3}{x^2-2x-8}$

SOLUTION:

Step 1 Find the zeros.

There is no zero.

Step 2 Find the asymptotes.

Find the vertical asymptote. Set $b(x) = 0$.

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

$x - 4 = 0$ or $x + 2 = 0$ — Zero Product Property

$x = 4$ or $x = -2$ — Solve each equation.

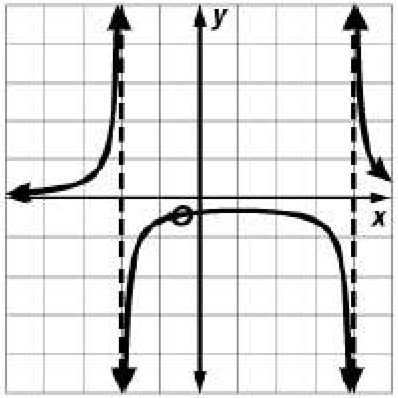
Because the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is the line $y = 0$.

Step 3 Draw the graph.

Graph the asymptote. Then make a table of values, and graph.

Step 3 Draw the graph.
Graph the asymptote. Then make a table of values, and graph.

x	$f(x)$
–5	$\frac{1}{9}$
–4	≈ 0.2
–3	≈ 0.4
–1	–0.6
0	≈ 0.38
1	$-\frac{1}{3}$
2	≈ –0.38
3	–0.6
5	≈ 0.4



28. $f(x) = \dfrac{x^3}{2x+2}$

SOLUTION:

Step 1 Find the zeros.
There is a zero at $x = 0$.

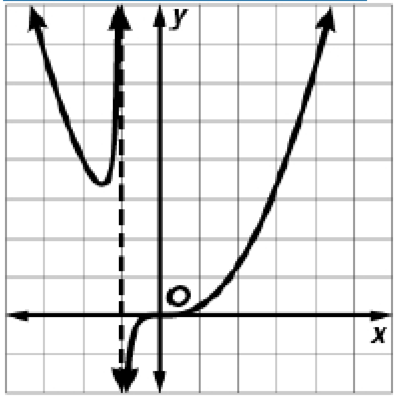
Step 2 Find the asymptotes.
Find the vertical asymptote. Set $b(x) = 0$.

$$2x + 2 = 0$$
$$2x = -2$$
$$x = -1$$

Because the degree of the numerator is greater than the degree of the denominator, there is no horizontal asymptote.

Step 3 Draw the graph.
Graph the asymptote. Then make a table of values, and graph.

x	$f(x)$
–5	≈ 15.6
–4	≈ 10.7
–3	6.75
–2	4
0	0
1	$\frac{1}{4}$
2	$1\frac{1}{3}$
3	≈ 3.4
4	6.4



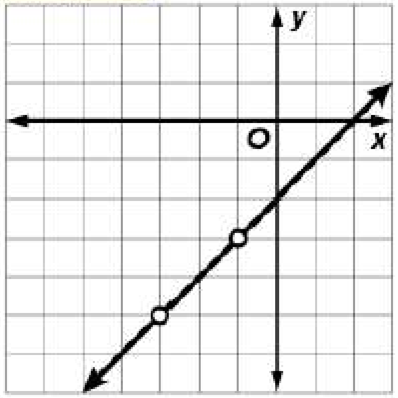
29. $f(x) = \dfrac{2x^3 + 4x^2 - 10x - 12}{2x^2 + 8x + 6}$

SOLUTION:

Notice that

$\frac{2x^3+4x^2-10x-12}{2x^2+8x+6} = $ [illegible] $= x - 2$.

However, because the denominator of the original function cannot be 0, there is a discontinuity at $x + 3 = 0$ or $x = -3$ and at $x + 1 = 0$, or $x = -1$. Therefore, the graph of $f(x) = \dfrac{2x^3 + 4x^2 - 10x - 12}{2x^2 + 8x + 6}$ is the graph of $f(x) = x - 2$ with a hole or point of discontinuity at $x = -3$ and $x = -1$.



18	Graph rational functions with oblique asymptotes and point discontinuity	11-30	344

Solution

30. $f(x) = \frac{(x+1)^2}{2x-1}$

SOLUTION:

Step 1 Find the zeros.

There is a zero at $x = -1$.

Step 2 Find the asymptotes.

Find the vertical asymptote. Set $b(x) = 0$.

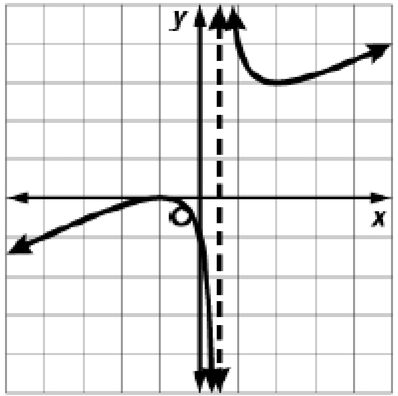
$$2x - 1 = 0$$
$$2x = 1$$
$$x = \frac{1}{2}$$

Because the degree of the numerator is greater than the degree of the denominator, there is no horizontal asymptote.

Step 3 Draw the graph.

Graph the asymptote. Then make a table of values, and graph.

x	$f(x)$
–4	–1
–3	≈ –0.6
–2	–0.2
–1	0
0	–1
1	4
2	3
3	3.2
5	4



19	Recognize and solve direct and joint variation problems	19-22	351

Example 4

19. Suppose a varies directly as b, and a varies inversely as c. Find b when $a = 5$ and $c = -4$, if $b = 12$ when $c = 3$ and $a = 8$.

20. Suppose x varies directly as y, and x varies inversely as z. Find z when $x = 10$ and $y = -7$, if $z = 20$ when $x = 6$ and $y = 14$.

19	Recognize and solve direct and joint variation problems	19-22	351

21. Suppose a varies directly as b, and a varies inversely as c. Find b when $a = 2.5$ and $c = 18$, if $b = 6$ when $c = 4$ and $a = 96$.

22. Suppose x varies directly as y, and x varies inversely as z. Find z when $x = 32$ and $y = 9$, if $z = 16$ when $x = 12$ and $y = 4$.

19	Recognize and solve direct and joint variation problems	19-22	351

solution method

Lesson 7-5

Variation

Learn Inverse Variation and Combined Variation

Two quantities, x and y, are related by an **inverse variation** if their product is equal to a constant k.

Key Concept • Inverse Variation

Words: y varies inversely as x if there is some nonzero constant k such that $xy = k$ or $y = \frac{k}{x}$, where $x \neq 0$ and $y \neq 0$.

Example: If $xy = 8$ and $x = 12$, then $y = \frac{8}{12}$ or $\frac{2}{3}$.

When y varies inversely as x and the constant proportionality k is positive, one quantity increases while the other decreases. You can use a proportion such as $\frac{x^1}{y^2} = \frac{x^2}{y^1}$ to solve inverse variation problems in which some quantities are known.

Combined variation occurs when one quantity varies directly and/or inversely as two or more other quantities.

If you know that y varies directly as x, that y varies inversely as z, and one set of values, you can use a proportion to find another set of corresponding values.

If $y_1 = \frac{kx_1}{z_1}$ and $y_2 = \frac{kx_2}{z_2}$, then $\frac{y_1z_1}{x_1} = k$ and $\frac{y_2z_2}{x_2} = k$.

Therefore, $\frac{y_1z_1}{x_1} = \frac{y_2z_2}{x_2}$.

Example 4 Combined Variation

20	Solve rational equations	1-12	361

Example 1

Solve each equation. Check your solutions.

1. $\frac{2x+3}{x+1} = \frac{3}{2}$

2. $\frac{-12}{y} = y - 7$

3. $\frac{9}{x-7} - \frac{7}{x-6} = \frac{13}{x^2 - 13x + 42}$

4. $\frac{13}{y+3} - \frac{12}{y+4} = \frac{18}{y^2 + 7y + 12}$

5. $\frac{14}{x-2} - \frac{18}{x+1} = \frac{22}{x^2 - x - 2}$

6. $\frac{2}{a+2} + \frac{10}{a+5} = \frac{36}{a^2 + 7a + 10}$

20	Solve rational equations	1-12	361

Example 2

7. $\frac{x}{2x-1} + \frac{3}{x+4} = \frac{21}{2x^2+7x-4}$

8. $\frac{2}{y-5} + \frac{y-1}{2y+1} = \frac{2}{2y^2-9y-5}$

9. $\frac{x-8}{2x+2} + \frac{x}{2x+2} = \frac{2x-3}{x+1}$

10. $\frac{12p+19}{p^2+7p+12} - \frac{3}{p+3} = \frac{5}{p+4}$

11. $\frac{2f}{f^2-4} + \frac{1}{f-2} = \frac{2}{f+2}$

12. $\frac{8}{t^2-9} + \frac{4}{t+3} = 1$

20	Solve rational equations	1-12	361

Solution

solution method

Lesson 7-6

Solving Rational Equations and Inequalities

Learn Solving Rational Equations

A **rational equation** contains at least one rational expression. To solve these equations, it is often easier to first eliminate the fractions. You can eliminate the fractions by multiplying each side of the equation by the least common denominator (LCD). Solving rational equations in this way can yield results that are not solutions of the original equation. You can identify these extraneous solutions by substituting each result into the original equation to see if it makes the equation true.

There are three types of problems that are commonly solved by using rational equations: mixture problems, uniform motion problems, and work problems.

Example 1 Solve a Rational Equation

Example 2 Solve a Rational Equation with an Extraneous Solution

Solve each equation. Check your solutions.

1. $\frac{2x+3}{x+1} = \frac{3}{2}$

SOLUTION:

The LCD for the terms is $2(x+1)$.

$$\frac{2x+3}{x+1} = \frac{3}{2}$$

$$2(x+1)\left(\frac{2x+3}{x+1}\right) = 2(x+1)\left(\frac{3}{2}\right)$$

$$2\cancel{(x+1)}\left(\frac{2x+3}{\cancel{x+1}}\right) = \cancel{2}(x+1)\left(\frac{3}{\cancel{2}}\right)$$

$$4x+6 = 3x+3$$

$$x+6 = 3$$

$$x = -3$$

ANSWER:

–3

2. $\frac{-12}{y} = y - 7$

SOLUTION:

The LCD for the terms is y.

$$\frac{-12}{y} = y - 7$$

$$y\left(\frac{-12}{y}\right) = y(y) - y(7)$$

$$\cancel{y}\left(\frac{-12}{\cancel{y}}\right) = y(y) - y(7)$$

$$-12 = y^2 - 7y$$

$$0 = y^2 - 7y + 12$$

$$0 = (y-4)(y-3)$$

$y - 4 = 0$ or $y - 3 = 0$ Zero Product

$y = 4$ or $y = 3$ Solve each e

ANSWER:

3, 4

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Solution

3. $\frac{9}{x-7} - \frac{7}{x-6} = \frac{13}{x^2-13x+42}$

SOLUTION:

The LCD for the terms is $(x-7)(x-6)$.

$$\frac{9}{x-7} - \frac{7}{x-6} = \frac{13}{x^2-13x+42}$$
$$(x-7)(x-6)\left(\frac{9}{x-7}\right) - (x-7)(x-6)\left(\frac{7}{x-6}\right) = (x-7)(x-6)\left(\frac{13}{x^2-13x+42}\right)$$
$$9x-54-7x+49=13$$
$$2x=18$$
$$x=9$$

ANSWER:

9

4. $\frac{13}{y+3} - \frac{12}{y+4} = \frac{18}{y^2+7y+12}$

SOLUTION:

The LCD for the terms is $(y+3)(y+4)$.

$$\frac{13}{y+3} - \frac{12}{y+4} = \frac{18}{y^2+7y+12}$$
$$(y+3)(y+4)\left(\frac{13}{y+3}\right) - (y+3)(y+4)\left(\frac{12}{y+4}\right) = (y+3)(y+4)\left(\frac{18}{y^2+7y+12}\right)$$
$$13y+52-12y-36=18$$
$$y=2$$

ANSWER:

2

5. $\frac{14}{x-2} - \frac{18}{x+1} = \frac{22}{x^2-x-2}$

SOLUTION:

The LCD for the terms is $(x-2)(x+1)$.

$$\frac{14}{x-2} - \frac{18}{x+1} = \frac{22}{x^2-x-2}$$
$$(x-2)(x+1)\left(\frac{14}{x-2}\right) - (x-2)(x+1)\left(\frac{18}{x+1}\right) = (x-2)(x+1)\left(\frac{22}{x^2-x-2}\right)$$
$$14x+14-18x+36=22$$
$$-4x=-28$$
$$x=7$$

ANSWER:

7

6. $\frac{2}{a+2} + \frac{10}{a+5} = \frac{36}{a^2+7a+10}$

SOLUTION:

The LCD for the terms is $(a+2)(a+5)$.

$$\frac{2}{a+2} + \frac{10}{a+5} = \frac{36}{a^2+7a+10}$$
$$(a+2)(a+5)\left(\frac{2}{a+2}\right) + (a+2)(a+5)\left(\frac{10}{a+5}\right) - (a+2)(a+5)\left(\frac{36}{a^2+7a+10}\right)$$
$$2a+10+10a+20=36$$
$$12a+30=36$$
$$12a=6$$
$$a=\frac{1}{2}$$

ANSWER:

$\frac{1}{2}$

7. $\frac{x}{2x-1} + \frac{3}{x+4} = \frac{21}{2x^2+7x-4}$

SOLUTION:

The LCD for the terms is $(2x-1)(x+4)$.

$$\frac{x}{2x-1} + \frac{3}{x+4} = \frac{21}{2x^2+7x-4}$$
$$(2x-1)(x+4)\left(\frac{x}{2x-1}\right) + (2x-1)(x+4)\left(\frac{3}{x+4}\right) = (2x-1)(x+4)\left(\frac{21}{2x^2+7x-4}\right)$$
$$x^2+4x+6x-3=21$$
$$x^2+10x-3=21$$
$$x^2+10x-24=0$$
$$(x+12)(x-2)=0$$

$x+12=0$ or $x-2=0$ Zero Product Property

$x=-12$ or $x=2$ Solve each equation.

Check each solution by substituting into the original equation.

Since neither solution results in a zero in the denominator, the solution is $x=-12$ and $x=2$.

ANSWER:

$-12, 2$

8. $\frac{2}{y-5} + \frac{y-1}{2y+1} = \frac{2}{2y^2-9y-5}$

SOLUTION:

The LCD for the terms is $(y-5)(2y+1)$.

$$\frac{2}{y-5} + \frac{y-1}{2y+1} = \frac{2}{2y^2-9y-5}$$
$$(y-5)(2y+1)\left(\frac{2}{y-5}\right) + (y-5)(2y+1)\left(\frac{y-1}{2y+1}\right) = (y-5)(2y+1)\left(\frac{2}{2y^2-9y-5}\right)$$
$$4y+2+y^2-6y+5=2$$
$$y^2-2y+5=0$$

There are no real number solutions, so the answer is the null set.

ANSWER:

Ø

9. $\frac{x-8}{2x+2} + \frac{x}{2x+2} = \frac{2x-3}{x+1}$

SOLUTION:

The LCD for the terms is $2(x+1)$.

$$\frac{x-8}{2x+2} + \frac{x}{2x+2} = \frac{2x-3}{x+1}$$
$$2(2x-1)\left(\frac{x-8}{2(x+1)}\right) + 2(2x-1)\left(\frac{x}{2(x+1)}\right) = 2(x+1)\left(\frac{2x-3}{x+1}\right)$$
$$x-8+x=4x-6$$
$$-8=2x-6$$
$$-2=2x$$
$$-1=x$$

Check each solution by substituting into the original equation.

Since the solution results in a zero in the denominator, the solution is the null set.

ANSWER:

Ø

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Solution

10. $\dfrac{12p+19}{p^2+7p+12}-\dfrac{3}{p+3}=\dfrac{5}{p+4}$

SOLUTION:

The LCD for the terms is $(p+3)(p+4)$.

$$\frac{12p+19}{p^2+7p+12}-\frac{3}{p+3}=\frac{5}{p+4}$$

$$\cancel{(p+3)(p+4)}\left(\frac{12p+19}{\cancel{p^2+7p+12}}\right)-\cancel{(p+3)}(p+4)\left(\frac{3}{\cancel{(p+3)}}\right)=(p+3)\cancel{(p+4)}\left(\frac{5}{\cancel{p+4}}\right)$$

$$\begin{aligned}12p+19-3p-12&=5p+15\\4p+7&=15\\4p&=8\\p&=2\end{aligned}$$

Check the solution by substituting into the original equation.

The solution is 2.

ANSWER:

2

11. $\dfrac{2f}{f^2-4}+\dfrac{1}{f-2}=\dfrac{2}{f+2}$

SOLUTION:

The LCD for the terms is $(f+2)(f-2)$.

$$\frac{2f}{f^2-4}+\frac{1}{f-2}=\frac{2}{f+2}$$

$$\cancel{(f+2)(f-2)}\left(\frac{2f}{\cancel{f^2-4}}\right)+\cancel{(f-2)}(f+2)\left(\frac{1}{\cancel{(f-2)}}\right)=(f-2)\cancel{(f+2)}\left(\frac{2}{\cancel{f+2}}\right)$$

$$\begin{aligned}2f+f+2&=2f-4\\f+2&=-4\\f&=-6\end{aligned}$$

Check the solution by substituting into the original equation.

The solution is –6.

ANSWER:

–6

12. $\dfrac{8}{t^2-9}+\dfrac{4}{t+3}=1$

SOLUTION:

The LCD for the terms is $(t+3)(t-3)$.

$$\frac{8}{t^2-9}+\frac{4}{t+3}=\frac{1}{1}$$

$$\cancel{(t+3)(t-3)}\left(\frac{8}{\cancel{t^2-9}}\right)+\cancel{(t+3)}(t-3)\left(\frac{4}{\cancel{(t+3)}}\right)=(t+3)(t-3)\left(\frac{1}{1}\right)$$

$$\begin{aligned}8+4t-12&=t^2-9\\-4&=t^2-4t-9\\0&=t^2-4t-5\end{aligned}$$

$t-5=0$ or $t+1=0$ Zero Product Property

$t=5$ or $t=-1$ Solve each equation.

Check the solution by substituting into the original equation.

The solutions are 5 and –1.

ANSWER:

5, –1