# 3-2 Logarithmic Functions

**Evaluate each expression.**

1. $\log_2 8$

*SOLUTION:*

$$\begin{aligned}\log_2 8 &= y\\ 2^y &= 8\\ 2^y &= 2^3\\ y &= 3\end{aligned}$$

2. $\log_{10} 10$

*SOLUTION:*

$$\begin{aligned}\log_{10} 10 &= y\\ 10^y &= 10\\ 10^y &= 10^1\\ y &= 1\end{aligned}$$

3. $\log_6 \frac{1}{36}$

*SOLUTION:*

$$\begin{aligned}\log_6 \frac{1}{36} &= y\\ 6^y &= \frac{1}{36}\\ 6^y &= 6^{-2}\\ y &= -2\end{aligned}$$

4. $4^{\log_4 1}$

*SOLUTION:*

$4^{\log_4 1} = 1$ Inverse Property of Logarithms

5. $\log_{11} 121$

*SOLUTION:*

$$\begin{aligned}\log_{11} 121 &= y\\ 11^y &= 121\\ 11^y &= 11^2\\ y &= 2\end{aligned}$$

6. $\log_2 2^3$

*SOLUTION:*

$$\begin{aligned}\log_2 2^3 &= y\\ 2^y &= 2^3\\ y &= 3\end{aligned}$$

7. $\log_{\sqrt{9}} 81$

*SOLUTION:*

$$\begin{aligned}\log_{\sqrt{9}} 81 &= y\\ \log_3 81 &= y\\ 3^y &= 81\\ 3^y &= 3^4\\ y &= 4\end{aligned}$$

8. log 0.01

*SOLUTION:*

$$\begin{aligned}\log 0.01 &= y\\ 10^y &= 0.01\\ 10^y &= 10^{-2}\\ y &= -2\end{aligned}$$

9. log 42

*SOLUTION:*

Enter log 42 in your calculator. ≈ 1.623

10. $\log_x x^2$

*SOLUTION:*

$$\begin{aligned}\log_x x^2 &= y\\ x^y &= x^2\\ y &= 2\end{aligned}$$

11. log 5275

*SOLUTION:*

Enter log 5275 in your calculator. ≈ 3.722

12. $\ln e^{-14}$

*SOLUTION:*

$$\begin{aligned}\ln e^{-14} &= -14\ln e\\ &= -14\end{aligned}$$

13. $3 \ln e^4$

*SOLUTION:*

$$\begin{aligned} 3\ln e^4 &= 3(4) \\ &= 12 \end{aligned}$$

14. $\ln (5 - \sqrt{6})$

*SOLUTION:*

Enter $\ln (5 - \sqrt{6})$ in your calculator. $\approx$ 0.936

15. $\log_{36} \sqrt[5]{6}$

*SOLUTION:*

$$\begin{aligned} \log_{36} \sqrt[5]{6} &= y \\ 36^y &= \sqrt[5]{6} \\ 36^y &= \sqrt[5]{\sqrt{36}} \\ 36^y &= \sqrt[10]{36} \\ 36^y &= 36^{\frac{1}{10}} \\ y &= \frac{1}{10} \end{aligned}$$

16. $4 \ln (7 - \sqrt{2})$

*SOLUTION:*

Enter $4 \ln (7 - \sqrt{2})$ in your calculator. $\approx$ 6.88

17. log 635

*SOLUTION:*

Enter log 635 in your calculator. $\approx$ 2.803

18. $\frac{\ln 2}{\ln 7}$

*SOLUTION:*

Enter $\frac{\ln 2}{\ln 7}$ in your calculator. $\approx$ 0.356

19. ln (–6)

*SOLUTION:*

The ln of a negative number is undefined.

20. $\ln\left(\frac{1}{e^{12}}\right)$

*SOLUTION:*

$$\begin{aligned}\ln\frac{1}{e^{12}} &= \ln e^{-12}\\ &= -12\ln e\\ &= -12\end{aligned}$$

21. ln 8

*SOLUTION:*

Enter ln 8 in your calculator. ≈ 2.079

22. $\log_{\sqrt[7]{4}} 64$

*SOLUTION:*

$$\begin{aligned}\log_{\sqrt[7]{4}} 64 &= y\\ \left(\sqrt[7]{4}\right)^y &= 64\\ \left(4^{\frac{1}{7}}\right)^y &= 4^3\\ 4^{\frac{y}{7}} &= 4^3\\ \frac{y}{7} &= 3\\ y &= 21\end{aligned}$$

23. $\frac{7}{lne}$

*SOLUTION:*

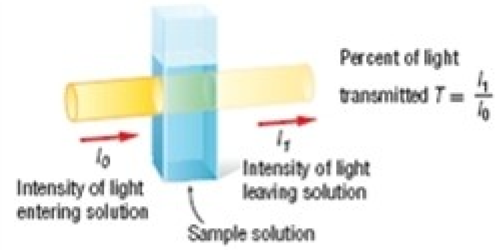
$$\begin{aligned}\frac{7}{\ln e} &= \frac{7}{1}\\ &= 7\end{aligned}$$

24. log 1000

*SOLUTION:*

$$\begin{aligned}\log 1000 &= \log 10^3\\ &= 3\log 10\\ &= 3\end{aligned}$$

25. **LIGHT** The amount of light $A$ absorbed by a solution is given by $A = 2 - \log 100T$, where $T$ is the percent of the light transmitted through the solution as shown in the diagram below.




In an experiment, a student shines light through two sample solutions containing different concentrations of a certain dye.

**a.** The percent of light transmitted through the first sample is 72%. How much light does the sample absorb to the nearest thousandth?

**b.** If the absorption of the second sample is 0.174, what percent of the light entering the solution is transmitted?

*SOLUTION:*

**a.**

$$\begin{aligned} A &= 2 - \log 100T \\ &= 2 - \log 100(0.72) \\ &= 2 - \log 72 \\ &\approx 0.14 \end{aligned}$$

**b.**

$$\begin{aligned} 0.174 &= 2 - \log 100T \\ \log 100T &= 2 - 0.174 \\ \log 100T &= 1.826 \\ 100T &= 10^{1.826} \\ T &= \frac{10^{1.826}}{100} \\ T &\approx 0.67 \text{ or } 67\% \end{aligned}$$

26. **SOUND** While testing the speakers for a concert, an audio engineer notices that the sound level reached a relative intensity of $2.1 \times 10^8$. The equation $D = \log I$ represents the loudness in decibels $D$ given the relative intensity $I$. What is the level of the loudness in decibels? Round to the nearest thousandth if necessary.

*SOLUTION:*

$$\begin{aligned} D &= \log I \\ &= \log\left(2.1 \times 10^8\right) \\ &\approx 8.322 \end{aligned}$$

## 3-2 Logarithmic Functions

27. **MEMORY** The students in Mrs. Ross' class were tested on exponential functions at the end of the chapter and then were retested monthly to determine the amount of information they retained. The average exam scores can be modeled by $f(x) = 85.9 - 9 \ln x$, where $x$ is the number of months since the initial exam. What was the average exam score after 3 months?

*SOLUTION:*

$$\begin{aligned} f(x) &= 85.9 - 9\ln x \\ f(3) &= 85.9 - 9\ln 3 \\ &\approx 76 \end{aligned}$$

**Sketch and analyze the graph of each function. Describe its domain, range, intercepts, asymptotes, end behavior, and where the function is increasing or decreasing.**

28. $f(x) = \log_4 x$

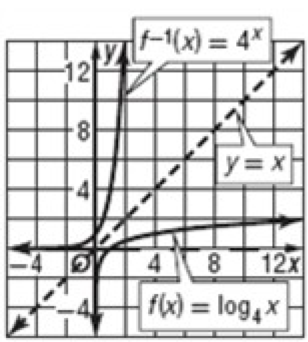
*SOLUTION:*

The inverse of $f(x) = \log_4 x$ is $f^{-1}(x) = 4^x$. Construct a table of values and sketch the graph of the inverse.

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $f^{-1}(x)$ | 0.015625 | 0.0625 | 0.25 | 1 | 4 | 16 | 64 |

Since the functions are inverses, you can obtain the graph of $f(x)$ by plotting $(f^{-1}(x), x)$.

| $x$ | 0.015625 | 0.0625 | 0.25 | 1 | 4 | 16 | 64 |
|---|---|---|---|---|---|---|---|
| $f^{-1}(x)$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |




D = $(0, \infty)$; R = $(-\infty, \infty)$; intercept: (1, 0); asymptote: $y$-axis; $\lim_{x\to 0} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$; increasing on $(0, \infty)$

# 3-2 Logarithmic Functions

29. $g(x) = \log_5 x$

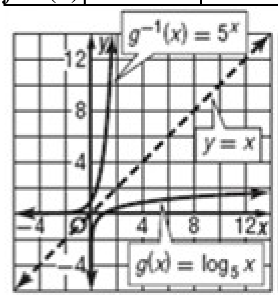
*SOLUTION:*

The inverse of $f(x) = \log_5 x$ is $f^{-1}(x) = 5^x$. Construct a table of values and sketch the graph of the inverse.

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $f^{-1}(x)$ | 0.008 | 0.04 | 0.2 | 1 | 5 | 25 | 125 |

Since the functions are inverses, you can obtain the graph of $f(x)$ by plotting $(f^{-1}(x), x)$.

| $x$ | 0.008 | 0.04 | 0.2 | 1 | 5 | 25 | 125 |
|---|---|---|---|---|---|---|---|
| $f^{-1}(x)$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |




D = (0, ∞ ); R = (– ∞ , ∞ ); intercept: (1, 0); asymptote: $y$-axis; $\lim_{x \to 0} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$; increasing on (0, ∞ )

30. $h(x) = \log_8 x$

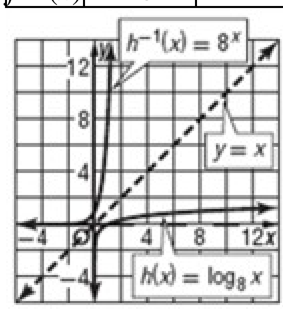
*SOLUTION:*

The inverse of $f(x) = \log_8 x$ is $f^{-1}(x) = 8^x$. Construct a table of values and sketch the graph of the inverse.

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $f^{-1}(x)$ | 0.002 | 0.016 | 0.125 | 1 | 8 | 64 | 512 |

Since the functions are inverses, you can obtain the graph of $f(x)$ by plotting $(f^{-1}(x), x)$.

| $x$ | 0.002 | 0.016 | 0.125 | 1 | 8 | 64 | 512 |
|---|---|---|---|---|---|---|---|
| $f^{-1}(x)$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |




D = (0, ∞ ); R = (– ∞ , ∞ ); intercept: (1, 0); asymptote: $y$-axis; $\lim_{x \to 0} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = \infty$; increasing on (0, ∞ )

# 3-2 Logarithmic Functions

31. $j(x) = \log_{\frac{1}{4}} x$

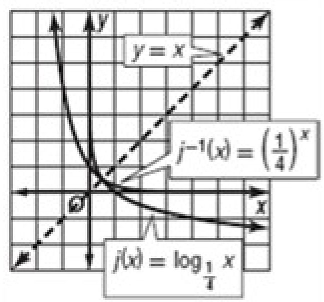
*SOLUTION:*

The inverse of $f(x) = \log_{0.25} x$ is $f^{-1}(x) = 0.25^x$. Construct a table of values and sketch the graph of the inverse.

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 |
|---|---|---|---|---|---|---|
| $f^{-1}(x)$ | 64 | 16 | 4 | 1 | 0.25 | 0.625 |

Since the functions are inverses, you can obtain the graph of $f(x)$ by plotting $(f^{-1}(x), x)$.

| $x$ | 64 | 16 | 4 | 1 | 0.25 | 0.625 |
|---|---|---|---|---|---|---|
| $f^{-1}(x)$ | −3 | −2 | −1 | 0 | 1 | 2 |




D = (0, ∞); R = (−∞, ∞); intercept: (1, 0); asymptote: $y$-axis; $\lim_{x \to 0} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = -\infty$; decreasing on (0, ∞)

32. $m(x) = \log_{\frac{1}{5}} x$

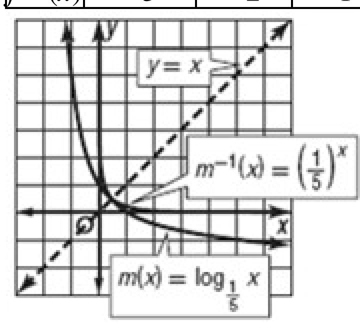
*SOLUTION:*

The inverse of $f(x) = \log_{0.2} x$ is $f^{-1}(x) = 0.2^x$. Construct a table of values and sketch the graph of the inverse.

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $f^{-1}(x)$ | 125 | 25 | 5 | 1 | 0.2 | 0.04 | 0.008 |

Since the functions are inverses, you can obtain the graph of $f(x)$ by plotting $(f^{-1}(x), x)$.

| $x$ | 125 | 25 | 5 | 1 | 0.2 | 0.04 | 0.008 |
|---|---|---|---|---|---|---|---|
| $f^{-1}(x)$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |




D = (0, ∞); R = (−∞, ∞); intercept: (1, 0); asymptote: $y$-axis; $\lim_{x \to 0} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = -\infty$; decreasing on (0, ∞)

33. $n(x) = \log_{\frac{1}{8}} x$

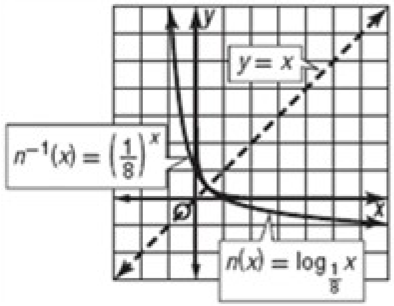
*SOLUTION:*

The inverse of $f(x) = \log_{0.125} x$ is $f^{-1}(x) = 0.125^x$. Construct a table of values and sketch the graph of the inverse.

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $f^{-1}(x)$ | 512 | 64 | 8 | 1 | 0.125 | 0.16 | 0.002 |

Since the functions are inverses, you can obtain the graph of $f(x)$ by plotting $(f^{-1}(x), x)$.

| $x$ | 512 | 64 | 8 | 1 | 0.125 | 0.16 | 0.002 |
|---|---|---|---|---|---|---|---|
| $f^{-1}(x)$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |



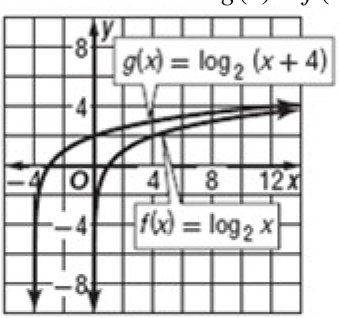

D = (0, ∞); R = (−∞, ∞); intercept: (1, 0); asymptote: $y$-axis; $\lim_{x \to 0} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = -\infty$; decreasing on (0, ∞)

**Use the graph of $f(x)$ to describe the transformation that results in the graph of $g(x)$. Then sketch the graphs of $f(x)$ and $g(x)$.**

34. $f(x) = \log_2 x$; $g(x) = \log_2 (x + 4)$

*SOLUTION:*

Find the relation of $g(x)$ to $f(x)$. $g(x) = f(x + 4)$, so the graph of $g(x)$ is the graph of $f(x)$ translated 4 units to the left.

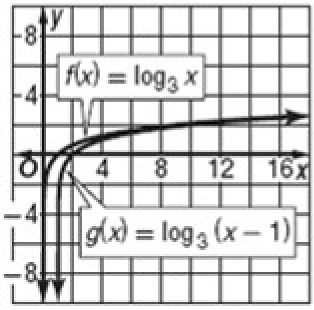


## 3-2 Logarithmic Functions

35. $f(x) = \log_3 x$; $g(x) = \log_3 (x - 1)$

*SOLUTION:*

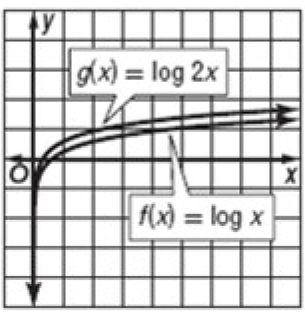
Find the relation of $g(x)$ to $f(x)$. $g(x) = f(x - 1)$, so the graph of $g(x)$ is the graph of $f(x)$ translated 1 unit to the right.




36. $f(x) = \log x$; $g(x) = \log 2x$

*SOLUTION:*

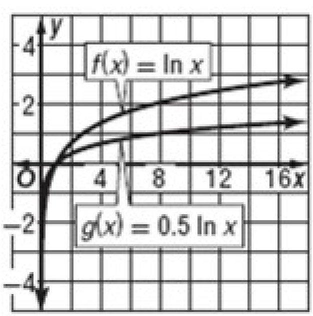
Find the relation of $g(x)$ to $f(x)$. $g(x) = f(2x)$, so the graph of $g(x)$ is the graph of $f(x)$ compressed horizontally by a factor of 2.




37. $f(x) = \ln x$; $g(x) = 0.5 \ln x$

*SOLUTION:*

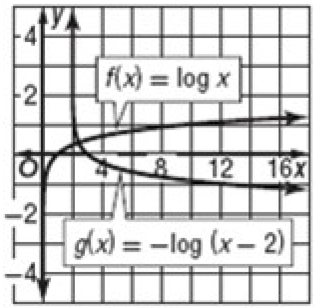
Find the relation of $g(x)$ to $f(x)$. $g(x) = 0.5 \cdot f(x)$, so the graph of $g(x)$ is the graph of $f(x)$ compressed vertically by a factor of 0.5.



38. $f(x) = \log x$; $g(x) = -\log (x - 2)$

*SOLUTION:*

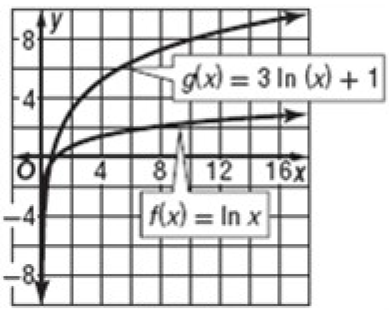
Find the relation of $g(x)$ to $f(x)$. $g(x) = -f(x - 2)$, so the graph of $g(x)$ is the graph of $f(x)$ reflected in the $x$-axis and translated 2 units to the right.




39. $f(x) = \ln x$; $g(x) = 3 \ln (x) + 1$

*SOLUTION:*

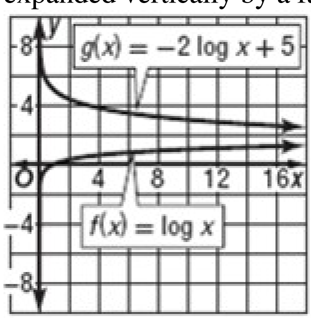
Find the relation of $g(x)$ to $f(x)$. $g(x) = 3 \cdot f(x) + 1$, so the graph of $g(x)$ is the graph of $f(x)$ expanded vertically by a factor of 3 and translated 1 unit up.




40. $f(x) = \log x$; $g(x) = -2 \log x + 5$

*SOLUTION:*

Find the relation of $g(x)$ to $f(x)$. $g(x) = -2 \cdot f(x) + 5$, so the graph of $g(x)$ is the graph of $f(x)$ reflected in the $x$-axis, expanded vertically by a factor of 2, and translated 5 units up.

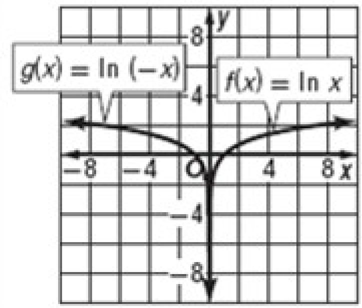


# 3-2 Logarithmic Functions

41. $f(x) = \ln x$; $g(x) = \ln(-x)$

*SOLUTION:*

Find the relation of $g(x)$ to $f(x)$. $g(x) = f(-x)$, so the graph of $g(x)$ is the graph of $f(x)$ reflected across the $y$-axis.




42. **INVESTING** The annual growth rate for an investment can be found using $r = \frac{1}{t} \ln \frac{P}{P0}$, where $r$ is the annual growth rate, $t$ is time in years, $P$ is the present value, and $P_0$ is the original investment. An investment of $10,000 was made in 2002 and had a value of $15,000 in 2009. What was the average annual growth rate of the investment?

*SOLUTION:*

$$\begin{aligned} r &= \frac{1}{t} \ln \frac{P}{P_0} \\ &= \frac{1}{7} \ln \frac{15,000}{10,000} \\ &\approx 0.058 \end{aligned}$$

**Determine the domain, range, $x$-intercept, and vertical asymptote of each function.**

43. $y = \log(x + 7)$

*SOLUTION:*

$y = \log(x + 7)$

The log of zero or a negative number is undefined and $x + 7 \leq 0$ when $x \leq -7$, so D = $\{x \mid x > -7, x \in \mathbb{R}\}$. The asymptote is at $x = -7$.

The range of the logarithmic function is R = $\{y \mid y \in \mathbb{R}\}$.

$\log 1 = 0$, so $y = 0$ when $x + 7 = 1$, or when $x = -6$.
$x$-intercept: $-6$

44. $y = \log x - 1$

*SOLUTION:*

$y = \log x - 1$

The log of zero or a negative number is undefined, so D = $\{x \mid x > 0, x \in \mathbb{R}\}$. The asymptote is at $x = 0$ or the $y$-axis.

The range of the logarithmic function is R = $\{y \mid y \in \mathbb{R}\}$.

$$\begin{aligned} y &= \log x - 1 \\ 0 &= \log x - 1 \\ 1 &= \log x \\ 10^1 &= x \\ 10 &= x \end{aligned}$$

$x$-intercept: 10

45. $y = \ln (x - 3)$

*SOLUTION:*

$y = \ln (x - 3)$

The ln of zero or a negative number is undefined and $x - 3 \leq 0$ when $x \leq 3$, so D = $\{x \mid x > 3, x \in \mathbb{R}\}$. The asymptote is at $x = 3$.

The range of the logarithmic function is R = $\{y \mid y \in \mathbb{R}\}$.

$$\begin{aligned} y &= \ln(x - 3) \\ 0 &= \ln(x - 3) \\ e^0 &= x - 3 \\ 1 &= x - 3 \\ 4 &= x \end{aligned}$$

$x$-intercept: 4

## 3-2 Logarithmic Functions

46. $y = \ln\left(x + \frac{1}{4}\right) - 3$

*SOLUTION:*

$y = \ln\left(x + \frac{1}{4}\right) - 3$

The ln of zero or a negative number is undefined and $x + \frac{1}{4} \leq 0$ when $x \leq -\frac{1}{4}$, so $D = \left\{x \middle| x > -\frac{1}{4}, x \in R\right\}$; The asymptote is at $x = -\frac{1}{4}$.

The range of the logarithmic function is $R = \{y \mid y \in R\}$.

$$y = \ln\left(x + \frac{1}{4}\right) - 3$$
$$0 = \ln\left(x + \frac{1}{4}\right) - 3$$
$$3 = \ln\left(x + \frac{1}{4}\right)$$
$$e^3 = x + \frac{1}{4}$$
$$e^3 - \frac{1}{4} = x$$
$$19.84 \approx x$$

$x$-intercept: 19.84

**Find the inverse of each equation.**

47. $y = e^{3x}$

*SOLUTION:*

$$y = e^{3x}$$
$$x = e^{3y}$$
$$\ln x = 3y$$
$$\frac{\ln x}{3} = y$$

48. $y = \log 2x$

*SOLUTION:*

$$y = \log 2x$$
$$x = \log 2y$$
$$10^x = 2y$$
$$\frac{10^x}{2} = y$$

49. $y = 4e^{2x}$

*SOLUTION:*

$$\begin{aligned} y &= 4e^{2x} \\ x &= 4e^{2y} \\ \frac{x}{4} &= e^{2y} \\ \ln\frac{x}{4} &= 2y \\ \frac{1}{2}\ln\frac{x}{4} &= y \end{aligned}$$

50. $y = 6 \log 0.5x$

*SOLUTION:*

$$\begin{aligned} y &= 6\log 0.5x \\ x &= 6\log 0.5y \\ \frac{x}{6} &= \log 0.5y \\ 10^{\frac{x}{6}} &= 0.5y \\ 2\cdot 10^{\frac{x}{6}} &= y \end{aligned}$$

51. $y = 20^x$

*SOLUTION:*

$$\begin{aligned} y &= 20^x \\ x &= 20^y \\ y &= \log_{20} x \end{aligned}$$

52. $y = 4(2^x)$

*SOLUTION:*

$$\begin{aligned} y &= 4(2^x) \\ x &= 4(2^y) \\ \frac{x}{4} &= 2^y \\ \log_2\frac{x}{4} &= y \end{aligned}$$

## 3-2 Logarithmic Functions

**Determine the domain and range of the inverse of each function.**

53. $y = \log x - 6$

*SOLUTION:*

$$y = \log x - 6$$
$$x = \log y - 6$$
$$x + 6 = \log y$$
$$10^{x+6} = y$$

D = $\{x \mid x \in \mathbb{R}\}$; R = $\{y \mid y > 0, y \in \mathbb{R}\}$

54. $y = 0.25e^{x+2}$

*SOLUTION:*

$$y = 0.25e^{x+2}$$
$$x = 0.25e^{y+2}$$
$$4x = e^{y+2}$$
$$\ln 4x = y + 2$$
$$\ln 4x - 2 = y$$

D = $\{x \mid x > 0, x \in \mathbb{R}\}$; R = $\{y \mid y \in \mathbb{R}\}$

55. **COMPUTERS** Gordon Moore, the cofounder of Intel, made a prediction in 1975 that is now known as Moore's Law. He predicted that the number of transistors on a computer processor at a given price point would double every two years.

**a.** Write Moore's Law for the predicted number of transmitters $P$ in terms of time in years $t$ and the initial number of transistors $I$.

**b.** In October 1985, a specific processor had 275,000 transistors. About how many years later would you expect the processor at the same price to have about 4.4 million transistors?

*SOLUTION:*

**a.** $P = I \cdot 2^{\frac{t}{2}}$

**b.**

$$P = I \cdot 2^{\frac{t}{2}}$$
$$4{,}400{,}000 = 275{,}000 \cdot 2^{\frac{t}{2}}$$
$$16 = 2^{\frac{t}{2}}$$
$$2^4 = 2^{\frac{t}{2}}$$
$$4 = \frac{t}{2}$$
$$8 = t$$

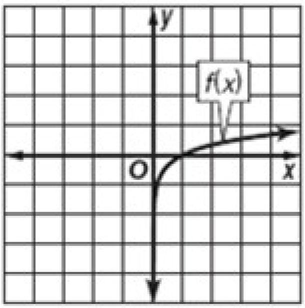
**Describe the domain, range, symmetry, continuity, and increasing/decreasing behavior for each logarithmic function with the given intercept and end behavior. Then sketch a graph of the function.**

56. $f(1) = 0; \lim_{x \to 0} f(x) = -\infty; \lim_{x \to \infty} f(x) = \infty$

*SOLUTION:*

The function is logarithmic, so R = (−∞, ∞). Logarithmic functions are continuous. Plot (1, 0).

Since the function approaches negative infinity as *x* approaches 0 and approaches infinity as *x* approaches infinity, and the function must pass through (1, 0), the function is increasing for (0, ∞) and the domain is D = (0, ∞).
sample answer:

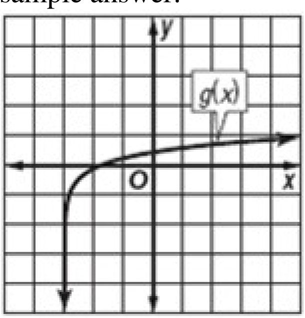



57. $g(-2) = 0; \lim_{x \to -3} g(x) = -\infty; \lim_{x \to \infty} g(x) = \infty$

*SOLUTION:*

The function is logarithmic, so R = (−∞, ∞). Logarithmic functions are continuous. Plot (−2, 0).

Since the function approaches negative infinity as *x* approaches −3 and approaches infinity as *x* approaches infinity, and the function must pass through (−2, 0), the function is increasing for (−3, ∞) and the domain is D = (−3, ∞).
sample answer:



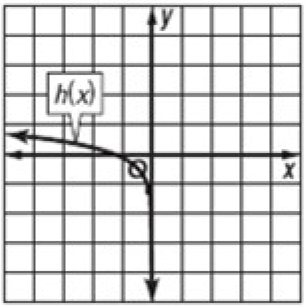
# 3-2 Logarithmic Functions

58. $h(-1) = 0; \lim_{x \to -\infty} h(x) = \infty; \lim_{x \to 0} h(x) = -\infty$

*SOLUTION:*

The function is logarithmic, so R = $(-\infty, \infty)$. Logarithmic functions are continuous. Plot (−1, 0).

Since the function approaches infinity as $x$ approaches negative infinity and approaches negative infinity as $x$ approaches zero, and the function must pass through (−1, 0), the function is decreasing for $(-\infty, 0)$ and the domain is D = $(-\infty, 0)$.
sample answer:

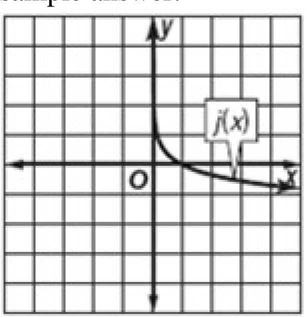



59. $j(1) = 0; \lim_{x \to 0} j(x) = \infty; \lim_{x \to \infty} j(x) = -\infty$

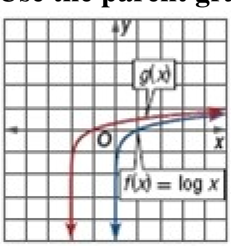
*SOLUTION:*

The function is logarithmic, so R = $(-\infty, \infty)$. Logarithmic functions are continuous. Plot (1, 0).

Since the function approaches negative infinity as $x$ approaches infinity and approaches infinity as $x$ approaches zero, and the function must pass through (1, 0), the function is decreasing for $(0, \infty)$ and the domain D = $(0, \infty)$.
sample answer:

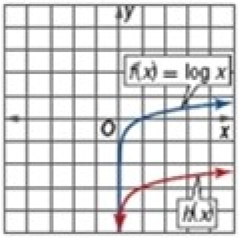



**Use the parent graph of $f(x) = \log x$ to find the equation of each function.**

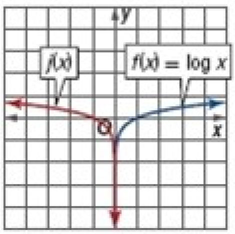
60. 


*SOLUTION:*

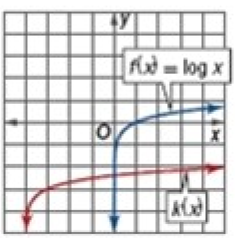
There is a translation 2 units left and no other transformation.
$g(x) = \log(x + 2)$

61. 


*SOLUTION:*
There is a translation 3 units down and no other transformation.
$h(x) = \log x - 3$

62. 


*SOLUTION:*
There is a reflection in the $y$-axis and no other transformation.
$j(x) = \log(-x)$

63. 


*SOLUTION:*
There are two translations, one 4 units left and one three units down.
$k(x) = \log(x + 4) - 3$

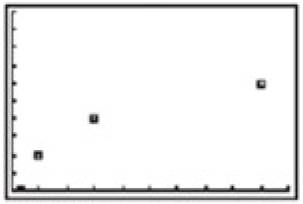
## 3-2 Logarithmic Functions

**GRAPHING CALCULATOR Create a scatter plot of the values shown in the table. Then use the graph to determine whether each statement is *true* or *false* .**

| x | 1 | 3 | 9 | 27 |
|---|---|---|---|---|
| y | 0 | 1 | 2 | 3 |

64. $y$ is an exponential function of $x$.

*SOLUTION:*

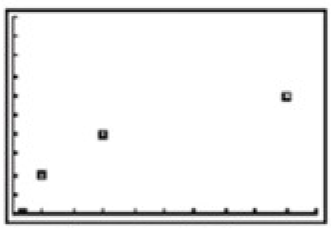


[0, 30] scl: 3 by [0, 5] scl: 0.5

The graph does not take on the shape of $y = a^x$.

65. $x$ is an exponential function of $y$.

*SOLUTION:*

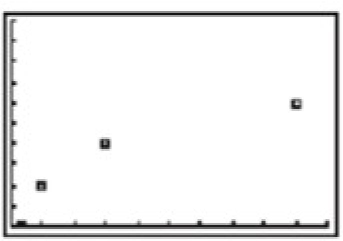


[0, 30] scl: 3 by [0, 5] scl: 0.5

The graph takes on the shape of $x = a^y$.

66. $y$ is a logarithmic function of $x$.

*SOLUTION:*

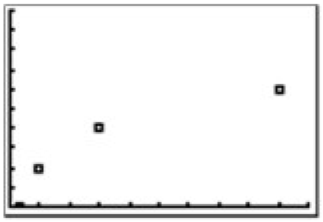


[0, 30] scl: 3 by [0, 5] scl: 0.5

The graph takes on the shape of $x = a^y$ or $y = \log_a x$.

67. $y$ is inversely proportional to $x$.

*SOLUTION:*



[0, 30] scl: 3 by [0, 5] scl: 0.5

The graph does not reflect $y = \frac{1}{x}$.

68. **BACTERIA** The function $t = \frac{InBInA}{2}$ models the amount of time $t$ in hours for a specific bacteria to reach amount $B$ from the initial amount $A$.

**a.** If the initial number of bacterial present is 750, how many hours would it take for the number of bacteria to reach 300,000? Round to the nearest hour.

**b.** Determine the average rate of change in bacteria per hour for the bacterial amounts in part **a**.

*SOLUTION:*

**a.**

$$t = \frac{\ln B - \ln A}{2}$$

$$t = \frac{\ln 300,000 - \ln 750}{2}$$

$$t \approx 3.00$$

**b.**

$$\frac{300,000 - 750}{3} = 99,750$$

69. **MULTIPLE REPRESENTATIONS** In this problem, you will compare the average rates of change for an exponential, a power, and a radical function.

**a. GRAPHICAL** Graph $f(x) = 2^x$ and $g(x) = x^2$ for $0 \le x \le 8$.

**b. ANALYTICAL** Find the average rate of change of each function from part **a** on the interval [4, 6].

**c. VERBAL** Compare the growth rates of the functions from part **a** as $x$ increases.

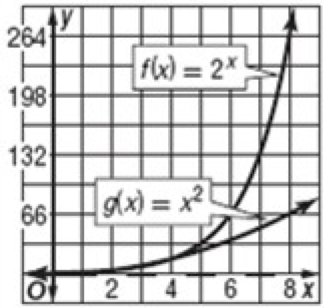
**d. GRAPHICAL** Graph $f(x) = \ln x$ and $g(x) = \sqrt{x}$.

**e. ANALYTICAL** Find the average rate of change of each function from part **d** on the interval [4, 6].

**f. VERBAL** Compare the growth rates of the functions from part **d** as $x$ increases.

*SOLUTION:*

**a.**




**b.**

$f(x)$

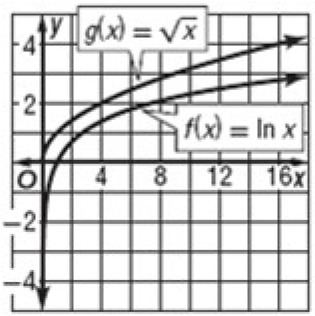
$$\frac{2^6 - 2^4}{6 - 4} = \frac{64 - 16}{2}$$
$$= 24$$

$g(x)$

$$\frac{6^2 - 4^2}{6 - 4} = \frac{36 - 16}{2}$$
$$= 10$$

**c.** Based on the graph and the answer to part **b**, the growth rate for an exponential function is greater than the growth rate of a power function as *x* increases.

**d.**




**e.**

$f(x)$

$$\frac{\ln 6 - \ln 4}{6 - 4} \approx 0.203$$

$g(x)$

$$\frac{\sqrt{6} - \sqrt{4}}{6 - 4} \approx 0.225$$

**f.** Based on the graph and the answer to part **d**, The growth rate for a root function is greater than the growth rate of the natural log function.

## 3-2 Logarithmic Functions

70. ***Writing in Math*** Compare and contrast the domain, range, intercepts, symmetry, continuity, increasing/decreasing behavior and end behavior of logarithmic functions with $a(x) = x^n$, $b(x) = x^{-1}$, $c(x) = a^x$, and $d(x) = e^x$.

*SOLUTION:*

Sample answer: The domain of the logarithmic parent function is $(0, \infty)$, whereas the domains of $c(x)$ and $d(x)$ are $(-\infty, \infty)$. The domains of $a(x)$ and $b(x)$ are $\{x| x \neq 0, x \in R\}$.The range of the parent logarithmic function is $(-\infty, \infty)$. The ranges of $a(x)$ and $b(x)$ are $\{y| y \neq 0, y \in R\}$. The ranges of $c(x)$ and $d(x)$ are $(0, \infty)$. The logarithmic parent function does not have a $y$-intercept. Likewise, $b(x)$ does not have a $y$-intercept. $a(x)$ has a $y$-intercept at the origin and $c(x)$ and $d(x)$ have $y$-intercepts at (0, 1). The logarithmic function, $a(x)$, $c(x)$, and $d(x)$ are not symmetrical, and $b(x)$ is symmetrical with respect to the origin. All of the functions are continuous except for $a(x)$ and $c(x)$ which are continuous only for $[0, \infty)$. The logarithmic parent function and $d(x)$ are increasing on the interval $(0, \infty)$. For $n > 0$, $a(x)$ is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$. For $n < 0$, $a(x)$ is decreasing on $(0, \infty)$ and increasing on $(-\infty, 0)$. $b(x)$ is decreasing on $(0, \infty)$ and increasing on $(-\infty, 0)$. For $a > 0$, $c(x)$ is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$. For $a < 0$, $c(x)$ is decreasing on $(0, \infty)$ and increasing on $(-\infty, 0)$. As the logarithmic parent function approaches 0, the graph approaches $-\infty$, and as it approaches $\infty$, the graph approaches $\infty$. For $n > 0$, $a(x)$ approaches $\infty$ as $x$ approaches $\infty$. $b(x)$ approaches 0 as $x$ approaches $-\infty$ and approaches 0 as $x$ approaches $\infty$ .For $a > 0$, $c(x)$ approaches 0 as $x$ approaches $-\infty$ and approaches $\infty$ as $x$ approaches $\infty$. $d(x)$ approaches 0 as $x$ approaches $-\infty$ and approaches $\infty$ as $x$ approaches $\infty$.

71. **REASONING** Explain why $b$ cannot be negative in $f(x) = \log_b x$.

*SOLUTION:*

Sample answer: The value of $b$ cannot be negative because even roots of negative numbers are undefined in the set of real numbers. For example, $\frac{1}{2} = \log_{-2} x$ can be rewritten as $(-2)^{\frac{1}{2}} = x$ by using the definition of a logarithm. The value of $(-2)^{\frac{1}{2}}$ is undefined.

72. **CHALLENGE** For $f(x) = \log_{10} (x - k)$, where $k$ is a constant, what are the coordinates of the $x$-intercept?

*SOLUTION:*

$$\begin{aligned} \log_{10}(x-k) &= 0 \\ 10^{x-k} &= 10^0 \\ x-k &= 1 \\ x &= k+1 \end{aligned}$$

73. ***Writing in Math*** Compare the large-scale behavior of exponential and logarithmic functions with base $b$ for $b = 2$, 6, and 10.

*SOLUTION:*

Sample answer: Exponential functions with larger bases increase more rapidly than with smaller bases. For example, $10^9 > 6^{10} > 2^{25}$. As the exponent increases, the difference between the $y$-values for each base will increase. Also, logarithmic functions with larger bases increase less rapidly than with smaller bases. For example, $\log_2 36 > \log_6 216 > \log_{10} 500$.

**REASONING Determine whether each statement is *true* or *false* .**

74. Logarithmic functions will always have a restriction on the domain.

*SOLUTION:*

False; sample answer: The graph of $y = \ln(x^2 + 1)$ does not have a vertical asymptote.

75. Logarithmic functions will never have a restriction on the range.

*SOLUTION:*

False; sample answer: The range of the graph of $y = \ln(x^2 + 1)$ is $y \mid y \geq 0$.

76. Graphs of logarithmic functions always have an asymptote.

*SOLUTION:*

False; sample answer: The graph of $y = \ln(x^2 + 1)$ does not have any asymptotes.

77. ***Writing in Math*** Use words, graphs, tables, and equations to compare logarithmic and exponential functions.

*SOLUTION:*

See students' work.

78. **AVIATION** When kerosene is purified to make jet fuel, pollutants are removed by passing the kerosene through a special clay filter. Suppose a filter is fitted in a pipe so that 15% of the impurities are removed for every foot that the kerosene travels.

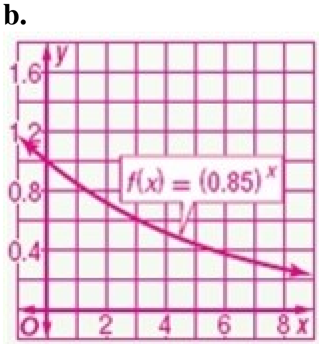
**a.** Write an exponential function to model the percent of impurity left after the kerosene travels $x$ feet.
**b.** Graph the function.
**c.** About what percent of the impurity remains after the kerosene travels 12 feet?
**d.** Will the impurities ever be completely removed? Explain.

*SOLUTION:*

**a.** For every foot, 15% is eliminated, so 85% remains.

$f(x) = (0.85)^x$.

**b.**




**c.**

$$f(12) = (0.85)^{12} \approx 0.14$$

**d.** No; the graph has an asymptote at $y = 0$, so the percent of impurities, $y$, will never reach 0.

**Solve each inequality.**

79. $x^2 - 3x - 2 > 8$

*SOLUTION:*

Simplify the inequality.

$$\begin{aligned} x^2 - 3x - 2 &> 8 \\ x^2 - 3x - 10 &> 0 \\ (x-5)(x+2) &> 0 \end{aligned}$$

Test $(-\infty, -2)$, $(-2, 5)$, and $(5, \infty)$.

$$\begin{aligned} &\text{for } (-\infty, -2),\ x = -4 \\ &(-4-5)(-4+2) = (-9)(-2) \\ &\qquad = 18 \\ &\qquad 18 > 0 \\ &\text{for } (-2, 5),\ x = 0 \\ &(0-5)(0+2) = (-5)(2) \\ &\qquad = -10 \\ &\qquad -10 \not> 0 \\ &\text{for } (5, \infty),\ x = 8 \\ &(8-5)(8+2) = (3)(10) \\ &\qquad = 30 \\ &\qquad 30 > 0 \end{aligned}$$

The solution is $(-\infty, -2)$ or $(5, \infty)$.

80. $4 \geq -(x-2)^3 + 3$

*SOLUTION:*

Simplify the inequality.

$$\begin{aligned} 4 &\geq -(x-2)^3 + 3 \\ 1 &\geq -(x-2)^3 \\ -1 &\leq (x-2)^3 \\ -1 &\leq x - 2 \\ 1 &\leq x \end{aligned}$$

The solution is $[1, \infty)$.

81. $\frac{2}{x} + 3 > \frac{29}{x}$

*SOLUTION:*

Simplify the inequality.

$$\begin{aligned}\frac{2}{x} + 3 &> \frac{29}{x}\\ 2 + 3x &> 29\\ 3x &> 27\\ x &> 9, x \neq 0\end{aligned}$$

Test $(-\infty, 0)$, $(0, 9)$, and $(9, \infty)$.

for $(-\infty, 0)$, $x = -2$

$$\begin{aligned}\frac{2}{-2} + 3 &> \frac{29}{-2}\\ -1 + 3 &> -14.5\\ 2 &> -14.5\end{aligned}$$

for $(0, 9)$, $x = 2$

$$\begin{aligned}\frac{2}{2} + 3 &> \frac{29}{2}\\ 1 + 3 &> 14.5\\ 4 &\not> 14.5\end{aligned}$$

for $(9, \infty)$, $x = 10$

$$\begin{aligned}\frac{2}{10} + 3 &> \frac{29}{10}\\ 0.2 + 3 &> 2.9\\ 3.2 &> 2.9\end{aligned}$$

The solution is $(-\infty, 0)$ or $(9, \infty)$.

82. $\frac{(x3)(x4)}{(x5)(x6)2} \le 0$

*SOLUTION:*

Simplify the inequality.

$$\frac{(x-3)(x-4)}{(x-5)(x-6)^2} \le 0, x \ne 5, 6$$

Test (−∞ , 3], [3, 4], [4, 5], and [5, ∞ ).

$$\text{for } (-\infty, 3], x = 1$$

$$\begin{aligned}\frac{(1-3)(1-4)}{(1-5)(1-6)^2} &= \frac{(-2)(-3)}{(-4)(-5)^2}\\ &= \frac{6}{-100}\\ &= -0.06\end{aligned}$$

$$-0.06 \le 0$$

$$\text{for } [3, 4], x = 3.5$$

$$\begin{aligned}\frac{(3.5-3)(3.5-4)}{(3.5-5)(3.5-6)^2} &= \frac{(0.5)(-0.5)}{(-1.5)(-2.5)^2}\\ &= \frac{-0.25}{-9.375}\\ &\approx 0.0267\end{aligned}$$

$$0.0267 \not\le 0$$

$$\text{for } [4, 5], x = 4.5$$

$$\begin{aligned}\frac{(4.5-3)(4.5-4)}{(4.5-5)(4.5-6)^2} &= \frac{(1.5)(0.5)}{(-0.5)(-1.5)^2}\\ &= \frac{0.75}{-1.125}\\ &\approx -0.67\end{aligned}$$

$$-0.67 \le 0$$

$$\text{for } [5, \infty], x = 7$$

$$\begin{aligned}\frac{(7-3)(7-4)}{(7-5)(7-6)^2} &= \frac{(4)(3)}{(2)(1)^2}\\ &= \frac{12}{2}\\ &= 6\end{aligned}$$

$$6 \not\le 0$$

The solution is (– ∞ , 3] or [4, 5].

83. $\sqrt{2x+3} - 4 \le 5$

*SOLUTION:*

Simplify the inequality.

$$\begin{aligned}\sqrt{2x+3} - 4 &\le 5\\ \sqrt{2x+3} &\le 9\\ 2x+3 &\le 81\\ 2x &\le 78\\ x &\le 39\end{aligned}$$

The value under the radical sign cannot be less than zero.

$$\begin{aligned}2x+3 &\ge 0\\ 2x &\ge -3\\ x &\ge -\frac{3}{2}\end{aligned}$$

The solution is $\left[-\frac{3}{2}, 39\right]$.

84. $\sqrt{x5} + \sqrt{x+7} \le 4$

*SOLUTION:*

Simplify the inequality.

$$\begin{aligned}\sqrt{x-5} + \sqrt{x+7} &\le 4\\ \sqrt{x-5} &\le 4 - \sqrt{x+7}\\ x-5 &\le 16 - 8\sqrt{x+7} + x + 7\\ x-5 &\le x + 23 - 8\sqrt{x+7}\\ 8\sqrt{x+7} &\le 28\\ \sqrt{x+7} &\le 3.5\\ x+7 &\le 12.25\\ x &\le 5.25\end{aligned}$$

The value under the radical sign cannot be less than zero, so $x$ must be greater than or equal to 5.

Test [5, 5.25].

$$\text{for } [5, 5.25],\ x = 5.1$$

$$\begin{aligned}\sqrt{5.1-5} + \sqrt{5.1+7} &= \sqrt{0.1} + \sqrt{12.1}\\ &\approx 3.79\\ 3.79 &\le 4\end{aligned}$$

The solution is [5, 5.25].

**Solve each equation.**

85. $\frac{2a-5}{a-9}+\frac{a}{a+9}=\frac{-6}{a^2-81}$

*SOLUTION:*

$$\begin{aligned}\frac{2a-5}{a-9}+\frac{a}{a+9}&=\frac{-6}{a^2-81}\\ \frac{(2a-5)(a+9)}{(a-9)(a+9)}+\frac{a(a-9)}{(a+9)(a-9)}&=\frac{-6}{(a+9)(a-9)}\\ (2a-5)(a+9)+a(a-9)&=-6\\ 2a^2+18a-5a-45+a^2-9a&=-6\\ 3a^2+4a-39&=0\\ (3a+13)(a-3)&=0\\ a&=-\frac{13}{3},3\end{aligned}$$

86. $\frac{2q}{2q+3}-\frac{2q}{2q-3}=1$

*SOLUTION:*

$$\begin{aligned}\frac{2q}{2q+3}-\frac{2q}{2q-3}&=1\\ \frac{2q(2q-3)}{(2q+3)(2q-3)}-\frac{2q(2q+3)}{(2q+3)(2q-3)}&=\frac{(2q+3)(2q-3)}{(2q+3)(2q-3)}\\ 2q(2q-3)-2q(2q+3)&=(2q+3)(2q-3)\\ 4q^2-6q-4q^2-6q&=4q^2-9\\ -12q&=4q^2-9\\ 0&=4q^2+12q-9\end{aligned}$$

Use the quadratic formula.

$$\begin{aligned}q&=\frac{-12\pm\sqrt{12^2-4(4)(-9)}}{2(4)}\\ &=\frac{-12\pm\sqrt{144+144}}{8}\\ &=\frac{-12\pm12\sqrt{2}}{8}\\ &=\frac{-3\pm3\sqrt{2}}{2}\end{aligned}$$

## 3-2 Logarithmic Functions

87. $\frac{4}{z-2} - \frac{z+6}{z+1} = 1$

*SOLUTION:*

$$\begin{aligned} \frac{4}{z-2} - \frac{z+6}{z+1} &= 1 \\ \frac{4(z+1)}{(z-2)(z+1)} - \frac{(z+6)(z-2)}{(z-2)(z+1)} &= \frac{(z-2)(z+1)}{(z-2)(z+1)} \\ 4(z+1) - (z+6)(z-2) &= (z-2)(z+1) \\ 4z+4-(z^2-2z+6z-12) &= z^2+z-2z-2 \\ -z^2+16 &= z^2-z-2 \\ 0 &= 2z^2-z-18 \end{aligned}$$

Use the quadratic formula.

$$\begin{aligned} z &= \frac{1 \pm \sqrt{(-1)^2-4(2)(-18)}}{2(2)} \\ &= \frac{1 \pm \sqrt{145}}{4} \end{aligned}$$

**Graph and analyze each function. Describe its domain, range, intercepts, end behavior, continuity, and where the function is increasing or decreasing.**

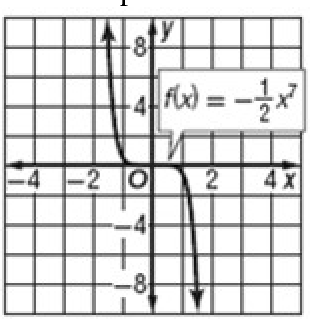
88. $f(x) = -\frac{1}{2}x^7$

*SOLUTION:*

Evaluate the function for several $x$-values in its domain.

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 1094 | 64 | 0.5 | 0 | −0.5 | −64 | −1094 |

Use these points to construct a graph.




The function is a monomial with an odd degree.

D = $(-\infty, \infty)$, R = $[-\infty, \infty)$; $x$-intercept: 0, $y$-intercept: 0; $\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = -\infty$; continuous for all real numbers; decreasing: $(-\infty, \infty)$

# 3-2 Logarithmic Functions

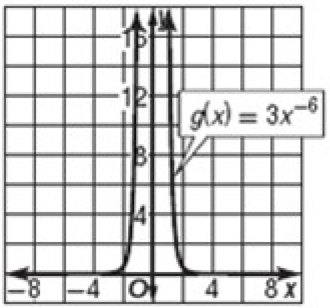
89. $g(x) = 3x^{-6}$

*SOLUTION:*

Evaluate the function for several $x$-values in its domain.

| $x$ | −3 | −2 | −1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.004 | 0.047 | 3 | undef. | 3 | 0.047 | 0.004 |

Use these points to construct a graph.




D = $(-\infty, 0)$ and $(0, \infty)$; R = $(0, \infty)$; no intercept; $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = 0$; infinite discontinuity at $x = 0$; increasing on $(-\infty, 0)$, decreasing on $(0, \infty)$

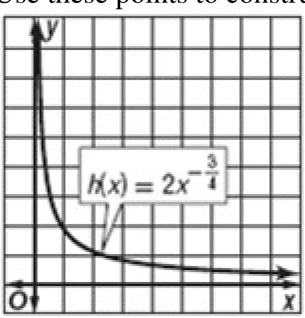
90. $h(x) = 2x^{-\frac{3}{4}}$

*SOLUTION:*

Evaluate the function for several $x$-values in its domain.

| $x$ | 0 | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | undef. | 3.36 | 2 | 1.48 | 1.19 | 1.01 | 0.88 |

Use these points to construct a graph.




D = $(0, \infty)$; R = $(0, \infty)$; no intercept; $\lim_{x \to 0} f(x) = \infty$ and $\lim_{x \to \infty} f(x) = 0$; continuous on $(0, \infty)$; decreasing on $(0, \infty)$

# 3-2 Logarithmic Functions

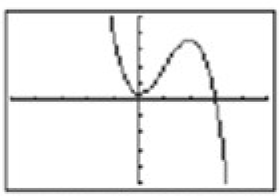
91. **MICROBIOLOGY** One model for the population $P$ of bacteria in a sample after $t$ days is given by

$$P(t) = 1000 - 19.75t + 20t^2 - \frac{1}{3}t^3.$$

**a.** What type of function is $P(t)$?
**b.** When is the bacteria population increasing?
**c.** When is it decreasing?

*SOLUTION:*

**a.** polynomial
**b.**
The function is a trinomial, so it will have a relative maximum and a relative minimum. Sketch a graph of the function.



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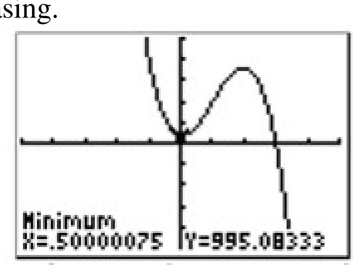
Locate the relative maximum and relative minimum to determine the intervals where the function is increasing and decreasing.

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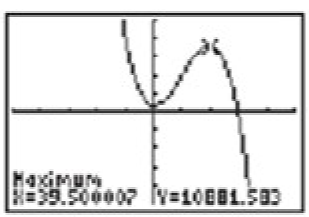


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The function is increasing on $0.5 < t < 39.5$.
c. The function is decreasing on $0 < t < 0.5$ and $t > 39.5$.

92. **SAT/ACT** The table below shows the per unit revenue and cost of three products at a sports equipment factory.

| Product | Revenue per Unit ($) | Cost per Unit ($) |
|---|---|---|
| football | $f$ | 4 |
| baseball | $b$ | 3 |
| soccer ball | 6 | $y$ |

If profit equals revenue minus cost, how much profit if they produce and sell two of each item?

**A** $2f + 2b - 2y - 2$
**B** $2y - 2b - 2f - 2$
**C** $f + b - y - 1$
**D** $b + 2f + y - 7$
**E** $2f + 2b - 2y - 26$

*SOLUTION:*

$$\begin{aligned} 2f - 2(4) + 2b - 2(3) + 2(6) - 2y &= 2f - 8 + 2b - 6 + 12 - 2y \\ &= 2f + 2b - 2y - 2 \end{aligned}$$

93. What is the value of $n$ if $\log_3 3^{4n-1} = 11$?

**F** 3
**G** 4
**H** 6
**J** 12

*SOLUTION:*

$$\begin{aligned} \log_3 3^{4n-1} &= 11 \\ 4n - 1 &= 11 \\ 4n &= 12 \\ n &= 3 \end{aligned}$$

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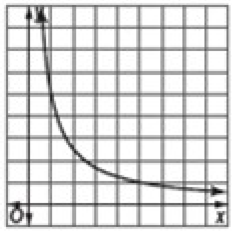
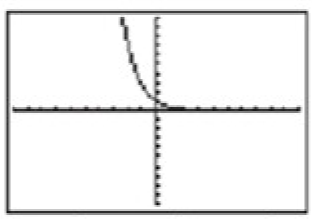
94. **REVIEW** The curve represents a portion of the graph of which function?



**A** $y = 50 - x$
**B** $y = \log x$
**C** $y = e^{-x}$
**D** $y = \frac{5}{x}$

*SOLUTION:*

Choice A is a linear function. Choice B is logarithmic and is shaped differently. The graph of choice C is shown below.



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This graph is not identical to the graph shown. The correct choice is D.

95. **REVIEW** A radioactive element decays over time according to $y = x\left(\frac{1}{4}\right)^{\frac{t}{200}}$, where $x$ = the number of grams present initially and $t$ = time in years. If 500 grams were present initially, how many grams will remain after 400 years?

**F** 12.5 grams
**G** 31.25 grams
**H** 62.5 grams
**J** 125 grams

*SOLUTION:*

$$\begin{aligned} y &= x\left(\frac{1}{4}\right)^{\frac{t}{200}} \\ &= 500\left(\frac{1}{4}\right)^{\frac{400}{200}} \\ &= 500\left(\frac{1}{4}\right)^{2} \\ &= 500 \cdot \frac{1}{16} \\ &= 31.25 \end{aligned}$$