Chapter 5

Applications of the Definite Integral

5.1 Area Between Curves

1. Area $= \int_1^3 \left[x^3 - \left(x^2 - 1\right)\right] dx$

$= \left(\frac{x^4}{4} - \frac{x^3}{3} + x\right)\Big|_1^3$

$= \left(\frac{81}{4} - \frac{27}{3} + 3\right) - \left(\frac{1}{4} - \frac{1}{3} + 1\right)$

$= \frac{160}{12} = \frac{40}{3}$

2. Area $= \int_0^2 [(x^2 + 2) - \cos x] dx$

$= \left(\frac{x^3}{3} + 2x - \sin x\right)\Big|_0^2 = \frac{20}{3} - \sin 2$

3. Area $= \int_{-2}^0 \left[e^x - (x - 1)\right] dx$

$= \left(e^x - \frac{x^2}{2} + x\right)\Big|_{-2}^0$

$= (1 - 0 + 0) - \left(e^{-2} - \frac{4}{2} + (-2)\right)$

$= 5 - e^{-2}$

4. Area $= \int_1^4 (x^2 - e^{-x}) dx$

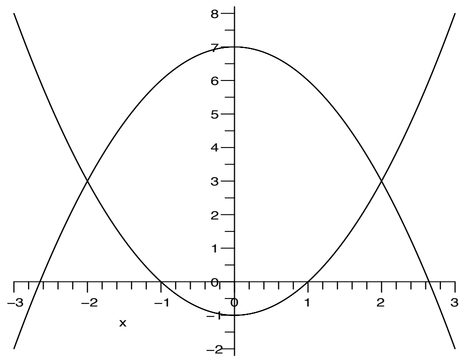
$= \left(\frac{x^3}{3} + e^{-x}\right)\Big|_1^4 = 21 + e^{-4} - e^{-1}$

5. Area $= \int_{-2}^2 \left[7 - x^2 - \left(x^2 - 1\right)\right] dx$

$= \left(8x - \frac{2x^3}{3}\right)\Big|_{-2}^2$

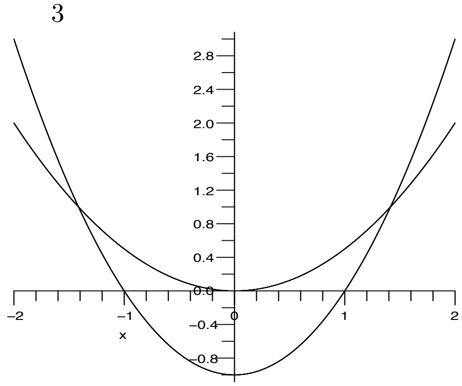
$= \left(16 - \frac{16}{3}\right) - \left(-16 + \frac{16}{3}\right)$

$= \frac{64}{3}$

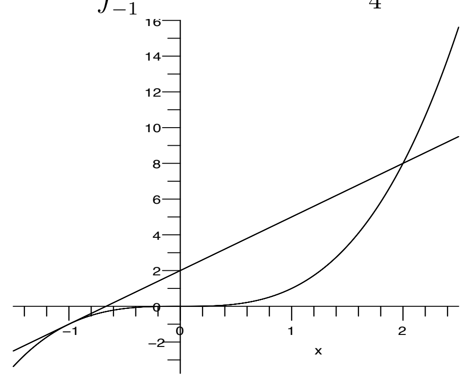


6. Area $= \int_{-\sqrt{2}}^{\sqrt{2}} \left[\frac{x^2}{2} - (x^2 - 1)\right] dx$

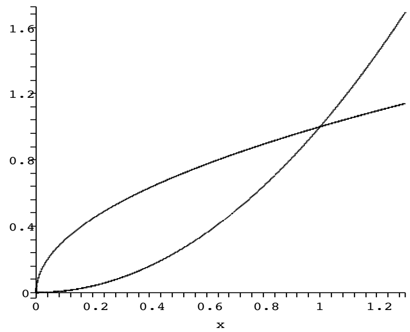
$= \frac{4\sqrt{2}}{3}$



7. Area $= \int_{-1}^2 (3x + 2 - x^3) dx = \frac{27}{4}$



8. Area $= \int_0^1 (\sqrt{x} - x^2) dx = \frac{1}{3}$

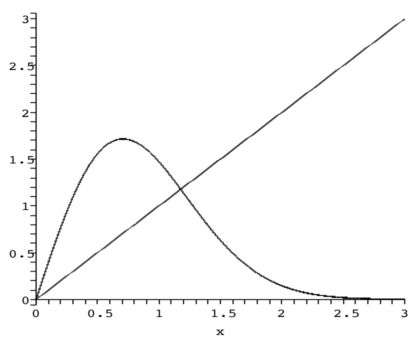


9. Area $= \int_0^{\sqrt{\ln 4}} \left(4xe^{-x^2} - x\right) dx$

$= -2e^{-x^2} - \dfrac{x^2}{2}\Big|_0^{\sqrt{\ln 4}}$

$= -2\left[\dfrac{1}{4} - 1\right] - \dfrac{\ln 4}{2}$

$= \dfrac{3 - \ln 4}{2}$



10. Area $= \int_{-1}^{0} \left(\dfrac{2}{x^2+1} + x\right) dx$

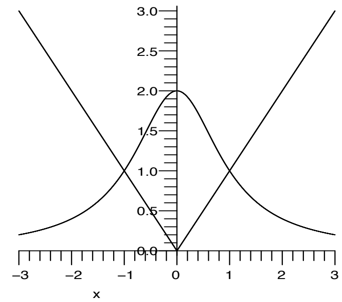
$+ \int_0^1 \left(\dfrac{2}{x^2+1} - x\right) dx$

$= (2\tan^{-1} x + \dfrac{x^2}{2})\Big|_{-1}^{0}$

$+(2\tan^{-1} x - \dfrac{x^2}{2})\Big|_0^1$

$= \left(\dfrac{\pi}{4} - \dfrac{1}{2}\right) + \left(\dfrac{\pi}{4} - \dfrac{1}{2}\right)$

$= \dfrac{\pi}{2} - 1$



11. Area $= \int_{-2}^{0} \left[x - \dfrac{5x}{x^2+1}\right] dx$

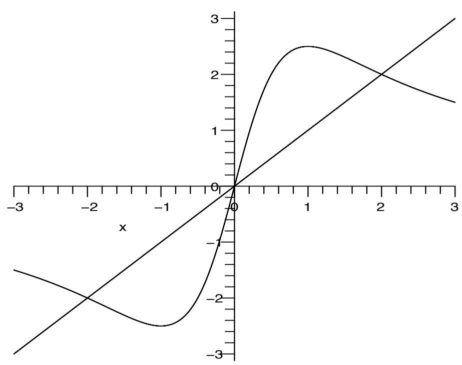
$+ \int_0^2 \left[\dfrac{5x}{x^2+1} - x\right] dx$

$= 2\int_0^2 \left[\dfrac{5x}{x^2+1} - x\right] dx$

$= 2\left(\dfrac{5}{2}\ln\left|x^2+1\right| - \dfrac{x^2}{2}\right)\Big|_0^2$

$= 5[\ln 5 - \ln 1] - [4 - 0]$

$= 5\ln 5 - 4$



12. Area $= \int_0^{\pi/4} (\cos x - \sin x) dx$

$+ \int_{\pi/4}^{5\pi/4} (\sin x - \cos x) dx$

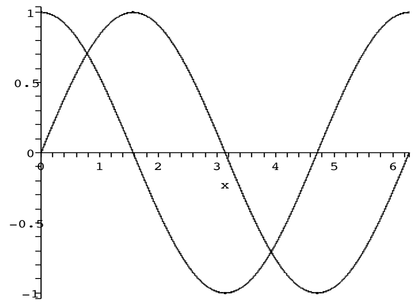
$+ \int_{5\pi/4}^{2\pi} (\cos x - \sin x) dx$

$= (\sin x + \cos x)|_0^{\pi/4}$

$+(-\cos x - \sin x)|_{\pi/4}^{5\pi/4}$

$+(\sin x + \cos x)|_{5\pi/4}^{2\pi}$

$= 4\sqrt{2}$

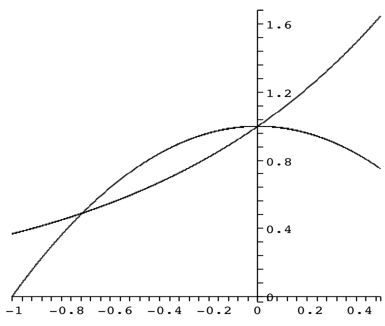


13. Area $= \displaystyle\int_{-.7145}^{0} (1-x^2) - e^x dx$

$= \left(-e^x + x - \dfrac{x^3}{3}\right)\Big|_{-.7145}^{0}$

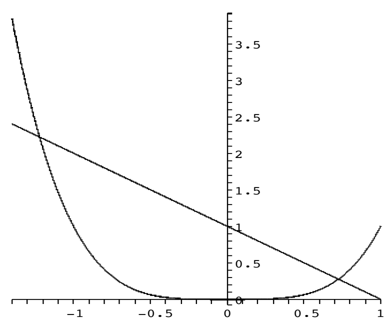
$= (-1+0-0) - (-1.08235)$

$= .08235$



14. Area $\approx \displaystyle\int_{-1.2207}^{0.72449} [(1-x) - x^4]dx$

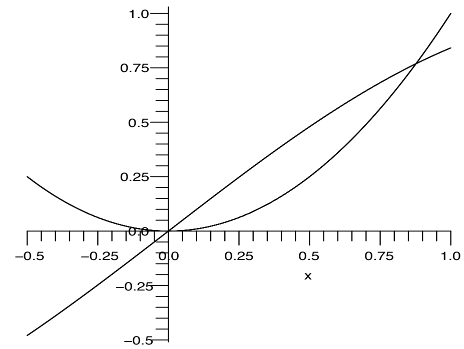
≈ 1.845787



15. Area $= \displaystyle\int_{0}^{.8767} \left(\sin x - x^2\right) dx$

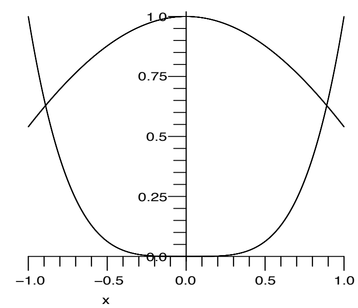
$= \left(-\cos x - \dfrac{x^3}{3}\right)\Big|_{0}^{.8767}$

$\approx .135697$



16. Area $\approx \displaystyle\int_{-.89055}^{0.89055} (\cos x - x^4)dx$

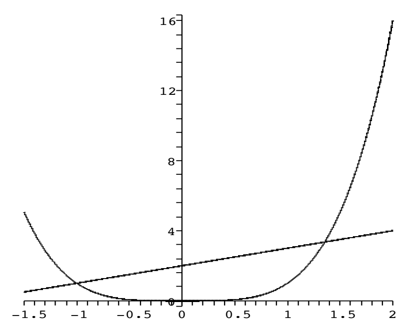
≈ 1.330782



17. Area $= \displaystyle\int_{-1}^{1.3532} \left(2 + x - x^4\right)dx$

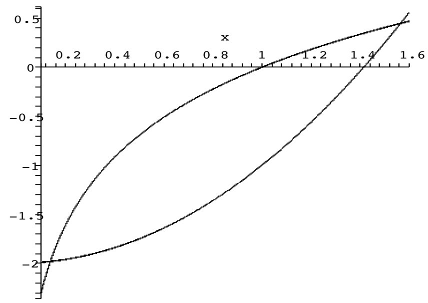
$= \left(2x + \dfrac{x^2}{2} - \dfrac{x^5}{5}\right)\Big|_{-1}^{1.3532}$

$= 4.01449$

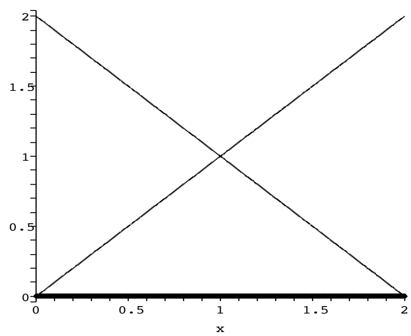


18. Area $\approx \displaystyle\int_{0.13793}^{1.5645} [\ln x - (x^2 - 2)]dx$

≈ 1.124448

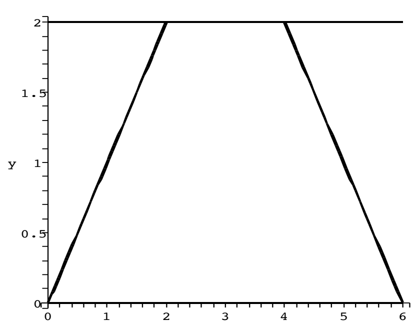


19. Area $= \int_0^1 [(2-y)-y]dy$
$= \int_0^1 [2-2y]dy$
$= \left(2y-y^2\right)\Big|_0^1$
$= 1-0=1$



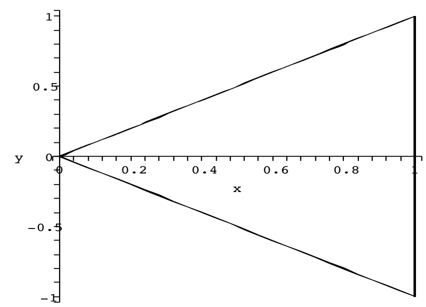
Area of triangle $= \frac{1}{2}(base)(height)$
$= \frac{1}{2}\cdot(2)\cdot(1)=1$

20. Area $= \int_0^2 [(6-y)-y]dy$
$= \int_0^2 (6-2y)dy$
$= \left(6y-y^2\right)\Big|_0^2$
$= (12-4)-(0-0)$
$= 8$



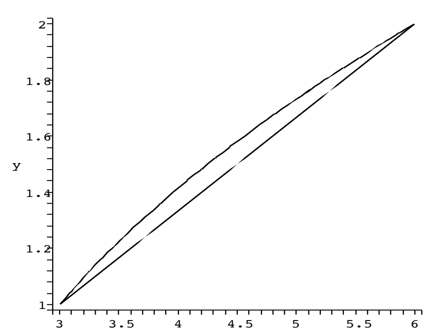
Area of Trapazium $= \frac{1}{2}(a+b)(h)$
$= \frac{1}{2}\cdot(8)\cdot(2)=8$

21. Area $= \int_0^1 [x-(-x)]dx$
$= 2\int_0^1 xdx = x^2\Big|_0^1$
$= 1-0=1$

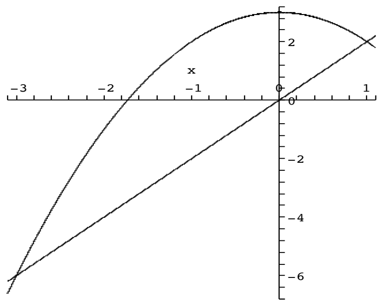


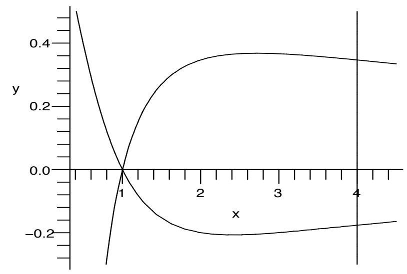
Area of triangle $= \frac{1}{2}(base)(height)$
$= \frac{1}{2}\cdot(2)\cdot(1)=1$

22. Area $= \int_1^2 \left[3y-\left(2+y^2\right)\right]dy$
$= \left(\frac{3}{2}y^2-2y-\frac{y^3}{3}\right)\Bigg|_1^2$
$= \left(6-4-\frac{8}{3}\right)-\left(\frac{3}{2}-2-\frac{1}{3}\right)$
$= \frac{1}{6}$

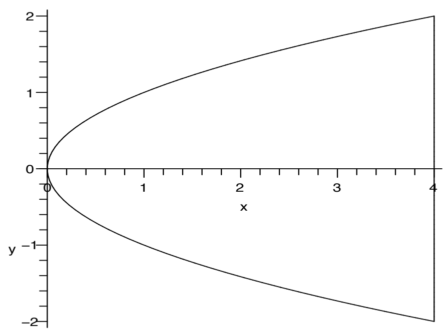


23. Area $= \int_{-3}^1 [(3-x^2)-2x]dx = \frac{32}{3}$

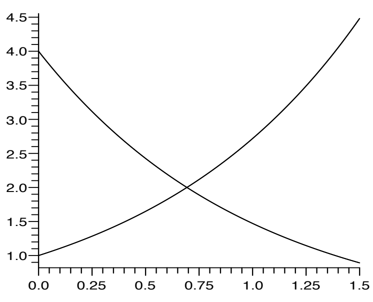




24. Area $= \displaystyle\int_{-2}^{2} (4 - y^2)dy = \frac{32}{3}$



25. Area $= \displaystyle\int_{0}^{\ln 2} (4e^{-x} - e^{x})dx \approx 1$



26. Area $= \displaystyle\int_{1}^{4} \left(\frac{\ln x}{x} - \frac{1-x}{x^2+1} \right) dx$

$$= \int_1^4 \frac{\ln x}{x} dx - \int_1^4 \frac{1}{x^2+1} dx$$

$$+ \frac{1}{2} \int_1^4 \frac{2x}{x^2+1}$$

$$= \left[\frac{\ln^2 x}{2} - \tan^{-1} x + \frac{1}{2} \ln |x^2 + 1| \right]_1^4$$

$$= \frac{\ln^4}{2} - \tan -14 + \frac{\ln 17}{2} + \frac{\pi}{4} - \frac{\ln 2}{2}$$

27. $\displaystyle\int_0^{.4} f_c(x) \approx \frac{.4}{3(4)} \{f_c(0) + 4f_c(.1)$

$+ 2f_c(.2) + 4f_c(.3) + f_c(.4)\} = 291.67$

$\displaystyle\int_0^{.4} f_e(x) \approx \frac{.4}{3(4)} \{f_e(0) + 4f_e(.1)$

$+ 2f_e(.2) + 4f_e(.3) + f_e(.4)\} = 102.33$

$$\frac{\int_0^{.4} f_c(x) - \int_0^{.4} f_e(x)}{\int_0^{.4} f_c(x)} \approx \frac{291.67 - 102.33}{291.67}$$

$= .6491 \ldots.$

$1 - .6491 = .3508,$

so the proportion of energy retained is about 35.08%.

28. Energy $= \dfrac{\int_0^m [f_c(x) - f_e(x)]dx}{\int_0^m f_c(x)dx}$

$$= \frac{\int_0^m f_c(x)dx}{\int_0^m f_c(x)dx} - \frac{\int_0^m f_e(x)dx}{\int_0^m f_c(x)dx}$$

$$= 1 - \frac{\int_0^m f_e(x)dx}{\int_0^m f_c(x)dx}$$

$\displaystyle\int_0^{0.18} f_c(x)dx$

$\approx \dfrac{0.045}{3} [f_c(0) + 4f_c(0.045) + 2f_c(0.09)$

$+ 4f_c(0.135) + f_c(0.18)]$

$= \dfrac{0.045}{3} [0 + 4(200) + 2(500) + 4(1000)$

$+ 1800]$

$= 114$

$\displaystyle\int_0^{0.18} f_e(x)dx$

$\approx \dfrac{0.045}{3} [f_e(0) + 4f_e(0.045) + 2f_e(0.09)$

$+ 4f_e(0.135) + f_e(0.18)]$

$= \dfrac{0.045}{3} (0 + 4(125) + 2(350) + 4(700)$

$+ 1800)$

$= 87$

Putting these together gives the proportion of energy lost as

Energy $\approx 1 - \dfrac{87}{114} \approx 0.2368.$

29. $\displaystyle\int_0^3 f_s(x) \approx \frac{3}{3(4)}\{f_s(0) + 4f_s(.75)$
$+ 2f_s(1.5) + 4f_s(2.25) + f_s(3)\} = 860$

$\displaystyle\int_0^3 f_r(x) \approx \frac{3}{3(4)}\{f_r(0) + 4f_r(.75)$
$+ 2f_r(1.5) + 4f_r(2.25) + f_r(3)\} = 800$

$1 - \left(\dfrac{860 - 800}{860}\right) = .9302$

Energy returned by the tendon is 93.02%.

30. As in Exercise 28, the proportion of energy returned by the arch is given by

$1 - \dfrac{\int_0^8 f_s(x)dx}{\int_0^8 f_r(x)dx}$

$\displaystyle\int_0^8 f_s(x)dx$

$\approx \dfrac{2}{3}[f_s(0) + 4f_s(2) + 2f_s(4) + 4f_s(6) + f_s(8)]$

$= \dfrac{2}{3}[0 + 4(300) + 2(1000) + 4(1800) + 3500]$

≈ 8366.67

$\displaystyle\int_0^8 f_r(x)dx$

$\approx \dfrac{2}{3}[f_r(0) + 4f_r(2) + 2f_r(4) + 4f_r(6) + f_r(8)]$

$= \dfrac{2}{3}[0 + 4(150) + 2(700) + 4(1300) + 3500]$

≈ 7133.33

Putting these together gives the proportion of energy lost as

Energy $\approx 1 - \dfrac{7133.33}{8366.67} \approx 0.1474.$

31. $A = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx = \frac{1}{3-0}\int_0^3 x^2 dx$

$= \left(\dfrac{1}{3} \cdot \dfrac{x^3}{3}\right)\Big|_0^3 = \dfrac{27}{9} - 0 = 3$

Relative to the interval $[0, 3]$, the inequality $x^2 < 3$ holds only on the subinterval $[0, \sqrt{3})$. We find

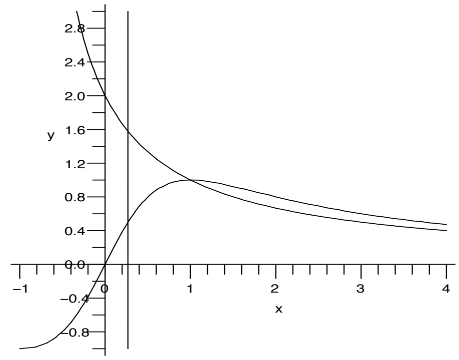
$\displaystyle\int_0^{\sqrt{3}} (3 - x^2)dx = \left(3x - \frac{x^3}{3}\right)\Big|_0^{\sqrt{3}}$

$= (3\sqrt{3} - \sqrt{3}) - (0 - 0)$
$= 2\sqrt{3},$ whereas

$\displaystyle\int_{\sqrt{3}}^3 (x^2 - 3)dx = \left(\frac{x^3}{3} - 3x\right)\Big|_{\sqrt{3}}^3$

$= (9 - 9) - (\sqrt{3} - 3\sqrt{3})$
$= 2\sqrt{3},$ the same.

32. Draw the graphs of the given functions,
$y = \dfrac{2}{(x+1)}$ and $y = \dfrac{2x}{(x^2+1)}$ for $x > 0$.



It may be observed from the graph that these functions cut each other at a single point at $x = 1$. From the graph it is observed that the curve $y = \dfrac{2}{(x+1)}$ lies above the curve $y = \dfrac{2x}{(x^2+1)}$ for $0 \le x \le 1$, for $x > 1$, $y = \dfrac{2x}{(x^2+1)}$ lies above the curve $y = \dfrac{2}{(x+1)}$

Let us find the area bounded by these curves between $x = 0$ and $x = 1$. It is given by

$\displaystyle\int_0^1 \left(\frac{2}{(x+1)} - \frac{2x}{(x^2+1)}\right)dx$

$= \left(\ln (x+1)^2 - \ln \left(x^2+1\right)\right)\Big|_0^1$

$= \ln 2 > \ln \left(\dfrac{3}{2}\right)$

$\Rightarrow 0 < t < 1$

Therefore

$\ln\left(\dfrac{3}{2}\right) = \displaystyle\int_0^t \left(\frac{2}{(x+1)} - \frac{2x}{(x^2+1)}\right)dx$

i.e. $\ln\left(\dfrac{3}{2}\right) = \left(\ln (x+1)^2 - \ln\left(x^2+1\right)\right)\Big|_0^t$

or $\ln\left(\dfrac{3}{2}\right) = \ln\left(\dfrac{(t+1)^2}{(t^2+1)}\right)$

$\Rightarrow 3t^2 + 3 = 2(t^2 + t + 1)$

i.e. $t = 2 \pm \sqrt{3}$

But as $0 < t < 1$, we consider $t = 2 - \sqrt{3}$

33. Let $y_1 = ax^2 + bx + c$, $y_2 = mx + n$, and $u = y_1 - y_2$. If we assume that $a < 0$, then $y_1 > y_2$ on (A, B) and the area between the curves is given by the integral

$\int_A^B (y_1 - y_2)dx$
$= \int_A^B udx = ux|_A^B - \int_A^B xdu.$

By assumption, u is zero ($y_1 = y_2$) at both A and B, so the first part of the last expression is zero. We must now show that
$-\int_A^B xdu = -\int_A^B x[2ax + (b-m)]dx$
is the same as
$|a|(B-A)^3/6$
$= |a|(B^3 - 3B^2A + 3BA^2 - A^3)/6.$

But again because $u = 0$ at both A and B, we know that
$aA^2 + bA + c = mA + n$ and
$aB^2 + bB + c = mB + n.$

By subtraction of the first from second, factoring out (and canceling) $B - A$, we learn $a(B + A) = m - b$, so that our target integral is also given by
$-2a \int_A^B x(x - \frac{A+B}{2})dx$
$= |a|\{2(B^3 - A^3)/3 - (A+B)(B^2 - A^2)/2\}$

and the student who cares enough can finish the details.

The case in which $a > 0 (y_2 > y_1)$ is not essentially different.

34. Perhaps the most straightforward way to handle this problem is by brute force. First, the area is given by
$$\text{Area } = \pm \int_A^B [(ax^3 + bx^2 + cx + d) - (kx^2 + mx + n)]dx$$
$$= \frac{a}{4}(A^4 - B^4) + \frac{(b-k)}{3}(B^3 - A^3) + \frac{(c-m)}{2}(B^2 - A^2) + (d-n)(B-A).$$

We can set up equations for the fact that the graphs meet at A and B. At A and B, we set the functions equal. At B, we set the derivatives equal.
$aA^3 + bA^2 + cA + d = kA^2 + mA + n$
$aB^3 + bB^2 + cB + d = kB^2 + mB + n$
$3aB^2 + 2bB + c = 2kB + m$

We now have a system of equations. We solve the last equation for m and plug the result in for m in the previous two equations. This transforms the three equations to
$aA^3 + (b-k)A^2 - 3aAB^2$
$- 2(b-k)AB + d - n = 0$
$- 2aB^3 - (b-k)B^2 + d - n = 0$
$m = 3aB^2 + 2(b-k)B + c.$

We solve the second equation for n and plug the result into the first equation which then gives
$aA^3 + (b-k)A^2 - 2(b-k)AB - 3aAB^2$
$+ 2aB^3 + (b-k)B^2 = 0$
$n = -2aB^3 - (b-k)B^2 + d$
$m = 3aB^2 + 2(b-k)B + c.$

Finally, solving the first equation for k gives
$k = aA + 2aB + b.$

We now substitute m, then n and then finally k in to the equation for area. After simplifying this finally gives
$\text{Area} = \frac{\pm a(A-B)^4}{12}.$

35. Let the upper parabola be
$y = y_1 = qx^2 + v + h$ and let the lower be
$y = y_2 = px^2 + v$. They are to meet at $x = w/2$, so we must have
$qw^2/4 + h = pw^2/4$, hence
$h = (p-q)w^2/4$ or $(q-p)w^2 = -4h.$

Using symmetry, the area between the curves is given by the integral
$2\int_0^{w/2} (y_1 - y_2)dx$
$= 2\int_0^{w/2} [h + (q-p)x^2]dx$
$= 2[hw/2 + (q-p)w^3/24]$
$= w[h + (q-p)w^2/12]$
$= w[h - 4h/12] = (2/3)wh.$

36. Solve the equation $2 - x^2 = mx$ we get
$x = \frac{m \pm \sqrt{m^2+8}}{2}$
So the area between $y = 2 - x^2$ and $y = mx$ is
$$\int_{(m-\sqrt{m^2+8})/2}^{(m+\sqrt{m^2+8})/2} (2 - x^2 - mx)dx$$
$$= \left(2x - \frac{x^3}{3} - \frac{mx^2}{2}\right)\Big|_{(m-\sqrt{m^2+8})/2}^{(m+\sqrt{m^2+8})/2}$$
$$= \frac{1}{6}(m^2+8)^{3/2}$$

The minimum of $(m^2+8)^{3/2}/6$ happens when $m = 0$ and then
$\frac{1}{6}(m^2+8)^{3/2} = \frac{1}{6} \cdot 8^{3/2} = \frac{8\sqrt{2}}{3}$

37. Solve for x in $x - x^2 = L$ we get
$x = \frac{1 \pm \sqrt{1-4L}}{2}$
$$A_1 = \int_0^{(1-\sqrt{1-4L})/2} [L - (x - x^2)]dx$$

$= \left(Lx - \frac{x^2}{2} + \frac{x^3}{3}\right)\Bigg|_0^{(1-\sqrt{1-4L})/2}$

$A_2 = \int_{(1-\sqrt{1-4L})/2}^{(1+\sqrt{1-4L})/2} [(x - x^2) - L]dx$

$= \left(\frac{x^2}{2} - \frac{x^3}{3} - Lx\right)\Bigg|_{(1-\sqrt{1-4L})/2}^{(1+\sqrt{1-4L})/2}$

By setting $A_1 = A_2$, we get the final answer $L = \frac{16}{3}$.

38. Solve for x in $x - x^2 = kx$ we get $x = 0, x = 1 - k$

And the areas are

$A_1 + A_2 = \int_0^1 (x - x^2)dx = \frac{1}{6}$

$A_2 = \int_0^{1-k} kxdx + \int_{1-k}^1 (x - x^2)dx$

$= \frac{kx^2}{2}\Bigg|_0^{1-k} + \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_{1-k}^1$

$= \frac{k(1-k)^2}{2} + \frac{1}{6} - \frac{(1-k)^2}{2} + \frac{(1-k)^3}{3}$

$= \frac{1}{6} \cdot [1 - (1-k)^3]$

We want $A_1 = A_2$, that is, we want $A_2 = 1/12$, that is,

$1 - (1-k)^3 = \frac{1}{2}$

$(1-k)^3 = \frac{1}{2}$

$1 - k = \frac{1}{\sqrt[3]{2}}$

$k = 1 - \frac{1}{\sqrt[3]{2}}$

39. (a) Consider $\int\limits_0^2 \left(2x - x^2\right) dx$

The integrand consists of the two curves $y = 2x$ and $y = x^2$. Both these curves intersect, when $2x = x^2$ i.e. when $x = 0$ or $x = 2$. therefore The given integral represents the area between the curves $y = 2x$ and $y = x^2$ Which is A_2.

(b) Consider $\int\limits_0^2 \left(4 - x^2\right) dx$

The integrand consists of two curves $y = 4$ and $y = x^2$. Both these curves intersect when $4 = x^2$ i.e. when $x = -2$ or $x = 2$. But we consider $x = 2$, as the area lies in the 1st Quadrant therefore the given integral represents the area between the curves $y = 4$ and $y = x^2$ which is $A_1 + A_2$.

(c) Consider $\int\limits_0^4 \left(2 - \sqrt{y}\right) dy$

Here the limits of integration correspond to the y-coordinates of the point of intersection of the two curves. This is because here the variable is y and not x. The integrand consists of two curves $x = 2$ and $x = \sqrt{y}$ $\left(\text{i.e.}\, y = x^2 \,\text{with}\, x > 0\right)$. Both these curves intersect, when $2 = \sqrt{y}$ i.e. when $y = 4$. therefore The given integral represents the area between the curves $x = 2$ and $x = \sqrt{y}$ which is A_3

(d) Consider $\int\limits_0^4 \left(\sqrt{y} - \frac{y}{2}\right) dy$

Here the limits of integration correspond to the y-coordinates of the point of intersection of the two curves. This is because here the variable is y and not x. The integrand consists of two curves $x = \sqrt{y}$ $\left(\text{i.e.}\, y = x^2 \,\text{with}\, x > 0\right)$ and $x = \frac{y}{2}$. Both these curves intersect, when $\frac{y}{2} = \sqrt{y}$ i.e. when $y^2 - 4y = 0$ i.e. at $y = 0$ and $y = 4$. therefore the given integral represents the area between the curves $x = \sqrt{y}$ and $x = \frac{y}{2}$ which is A_2(same as part **(a)**).

40. (a) Consider the area $A_2 + A_3$. It may be observed from the part (a) of the Exercise 39 that, A_2 is the area bounded by the curves $y = 2x$, $y = x^2$ between the ordinates $x = 0$ and $x = 2$. It may also be observed from the part (c) of the Exercise 39 that, A_3 is the area bounded by the curves $x = 2$ and $y = x^2$ i.e. $x = \sqrt{y}$ therefore from the given figure $A_2 + A_3$ is the area bounded by the curves $y = 2x$ i.e. $x = \frac{y}{2}$ and $x = 2$. therefore

$$A_2 + A_3 = \int\limits_0^4 \left(2 - \frac{y}{2}\right) dy.$$

Note that here we have y as the variable.

(b) Consider the area $A_1 + A_2$, refer part (b) of the Exercise 39 It is in fact the converse of that part.

(c) Consider the area A_1, from the given figure it may be observed that, A_1 is the area bounded by curves $y = 4$ and $y = 2x$. Between the ordinates $x = 0$ and $x = 2$. Therefore $A_1 = \int\limits_0^2 (4 - 2x)\, dx$

(d) A_3 refer part (c) or the Exercise 39. Note that here we have y as the variable.

41. The area between two curves $y = \sin^2(x)$ and $y = 1$, for $0 \le x \le t$ is given by:
$f(t) = \int_0^t (1 - \sin^2 x)\,dx = \int_0^t (\cos^2 x)\,dx$
$= \frac{1}{2}\int_0^t (1 + \cos 2x)\,dx$
$= \frac{1}{2}[x]_0^t + \frac{1}{4}[\sin 2x]_0^t$
$\Rightarrow f(t) = \frac{1}{2}t + \frac{1}{4}\sin 2t$
For finding the critical points,
$f'(t) = 0,$ therefore
$\frac{1}{2} + \frac{1}{4}\cos 2t \cdot (2) = 0.$
$\Rightarrow 1 + \cos 2t = 0$
or $\cos 2t = -1$
$\Rightarrow 2t = n\pi$ for $n = 1, 3, 5, ...$
or $t = \frac{n\pi}{2}$ for $n = 1, 3, 5, ...$
Now, $f''(t) = -\sin 2t$ substituting the value of t in $f''(t)$, we get $f''(t) = 0$. Therefore, $t = \frac{n\pi}{2}$ for $n = 1, 3, 5, ...$ are the points of inflection.

42. Given $g(x)$ is a continuous function of x, for $x \ge 0$ and $|g(x)| \le 1$. $f(t)$ is the area between $y = g(x)$ and $y = 1$ for $0 \le x \le t$, therefore $f(t) = \int_0^t (1 - g(x))\,dx$. As $g(x)$ has the local maxima at $x = a$, $g'(a) = 0$ and $g''(a) < 0$.
Now from (1)
$f'(t) = (1 - g(t))$
$\Rightarrow f''(t) = -g'(t)$
$\Rightarrow f''(a) = -g'(a) = 0$
also $f'(a) = (1 - g(a)) \ge 0$.
Thus $f(t)$ has an point of inflection at $x = a$ and a need not be the critical point, it is only if $g(a) = 1$. If there is a local minima at $x = a$, then $g'(a) = 0$ and $g''(a) > 0$. This does not affect the answer.

43. $f(4) = 16.1e^{.07(4)} = 21.3$
$g(4) = 21.3e^{.04(4-4)} = 21.3$
21.3 represents the consumption rate (million barrels per year) at time $t = 4$ (1/1/74).
$\int_4^{10} \left(16.1e^{.07t} - 21.3e^{.04(t-4)}\right) dt$
$= \left(230e^{.07t} - 532.5e^{.04(t-4)}\right)\Big|_4^{10}$
$= 14.4$ million barrels saved

44. Area $= \int_0^{10} [76e^{0.03t} - (50 - 6e^{0.09t})]\,dt$
≈ 483.616
This area represents amount of wood used by firewood that was not replaced with new growth.

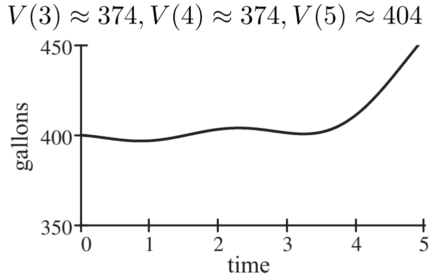
45. For $t \ge 0$,
$b(t) = 2e^{.04t} \ge 2e^{.02t} = d(t)$
$\int_0^{10} \left(2e^{.04t} - 2e^{.02t}\right) dt$
$= \left(50e^{.04t} - 100e^{.02t}\right)\Big|_0^{10}$
$= 2.45$ million people.
This number represents births minus deaths, hence population growth over the ten-year interval.

46. These curves intersect when
$T = \frac{\ln 3 - \ln 2}{.02} \approx 20.27325541$
The area between the curves for $0 \le t \le T$ is the decrease in population from $0 \le t \le T$ (because $b(t) < d(t)$ in this time period).
The area between the curves for $T \le t \le 30$ is the increase in population from $T \le t \le 30$ (because $b(t) > d(t)$ in this time period).
The change in population is given by the integral:
$\Delta P = \int_0^3 [b(t) - d(t))]\,dt$
$= \int_0^3 2e^{0.04t} - 4e^{0.02t}\,dt$
≈ 7.3120 million people

47. Without formulae or tables, only rough or qualitative estimates are possible.

time	1	2	3	4	5
amount	397	403	401	412	455

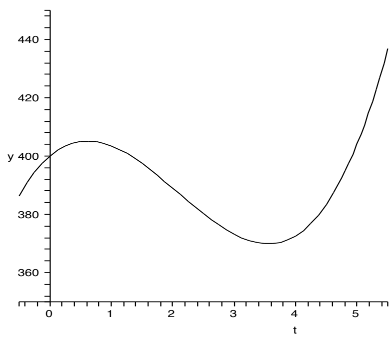
$V(3) \approx 374, V(4) \approx 374, V(5) \approx 404$



48. The change in amount of water is equal to the integral of the difference between the functions (the rate in minus the rate out). Approximating this integral:
$\int_0^1 (\text{Into} - \text{Out})\,dt \approx 0$

$$\int_0^2 (\text{Into} - \text{Out})\, dt \approx -8$$
$$\int_0^3 (\text{Into} - \text{Out})\, dt \approx -26$$
$$\int_0^4 (\text{Into} - \text{Out})\, dt \approx -26$$
$$\int_0^5 (\text{Into} - \text{Out})\, dt \approx 4$$

Therefore $V(1) = 400$, $V(2) \approx 392$, $V(3) \approx 374$, $V(4) \approx 374$, $V(5) \approx 404$.



49. In this set-up, p is price and q is quantity. We find that $D(q) = S(q)$ only if $D(q) = S(q)$.

$$10 - \frac{q}{40} = 2 + \frac{q}{120} + \frac{q^2}{1200}$$
$$12000 - 30q = 2400 + 10q + q^2$$
$$q^2 + 40q - 9600 = 0$$
$$(q - 80)(q + 120) = 0$$

within the range of the picture only at $q = 80$. Thus $q^* = 80$ and $p^* = D(q^*) = S(q^*) = 8$.

Consumer surplus, as an area, is that part of the picture below the D curve, above $p = p^*$, and to the left of $Q = q^*$.

Numerically in this case the consumer surplus is

$$\int_0^{q^*} [D(q) - p^*]\, dq = \int_0^{80} \left(2 - \frac{q}{40}\right) dq$$
$$= 2q - \frac{q^2}{80}\bigg|_0^{80} = 160 - 80 = 80.$$

The units are dollars (q counting items, p in dollars per item).

50. The intersection point is approximately $(q^*, p^*) = (76, 8)$. Therefore

$$PS = p^*q^* - \int_0^{q^*} S(q)\, dq$$
$$= (8)(76) - \int_0^{76} \left(2 + \frac{q}{120} + \frac{q^2}{1200}\right) dx$$
$$= \frac{86849}{225} \approx 386.00.$$

51. The curves, meeting as they do at 2 and 5, represent the derivatives C' and R'. The area (a) between the curves over the interval $[0, 2]$ is the loss resulting from the production of the first 2000 items. The area (b) between the curves over the interval $[2, 5]$ is the profit resulting from the production of the next 3000 items. The area (c), as the *sum* of the two previous (call it $(a) + (b)$), is *without meaning.* However, the *difference* $(b) - (a)$ would be the total profit on the first 5000 items, or, if negative, would represent the loss. The area (d) between the curves over the interval $[5, 6]$ represents the loss attributable to the (unprofitable) production of the next thousand items after the first 5000.

52. Profit increases when revenue is larger than cost. The point $x = 2$ represents a local minimum in profit. The point $x = 5$ represents a local maximum in profit.

5.2 Volume: Slicing, Disks and Washers

1. $V = \int_{-1}^{3} A(x)dx = \int_{-1}^{3} (x + 2)dx$

$= \left(\frac{x^2}{2} + 2x\right)\bigg|_{-1}^{3} = \left(\frac{9}{2} + 6\right) - \left(\frac{1}{2} - 2\right)$

$= 12$

2. $V = \int_0^{10} 10e^{0.01x}dx = \left(1000e^{0.01x}\right)\Big|_0^{10}$

$= 1000(e^{0.1} - 1)$

3. $V = \pi \int_0^2 (4 - x)^2 dx = -\frac{\pi}{3}(4 - x)^3\Big|_0^2$

$= -\frac{\pi}{3}(8 - 64) = \frac{56\pi}{3}$

4. $V = \int_1^4 2(x + 1)^2 dx$

$= \int_1^4 (2x^2 + 4x + 2)dx = 78$

5. (a) $f(0) = 750, f(500) = 0$

$f(x) = -\frac{75}{50}x + 750$

$V = \int_0^{500} \left(-\frac{75}{50}x + 750\right)^2 dx$

$= \frac{50}{75} \cdot \left(\frac{750^3}{3} - 0\right) = 93,750,000 \text{ ft}^3$

(b) In this case, essentially the same integral is set up as in Part (a):
$$V = \int_0^{250} \left(\frac{750}{500}\right)^2 (500-y)^2 dy$$
$$= 82,031,250 \text{ cubic feet}$$

6. $f(0) = 300, f(160) = 0$
$$f(x) = -\frac{15}{8}x + 300$$
$$V = \int_0^1 60\left(-\frac{15}{8}x + 300\right)^2 dx$$
$$= \frac{8}{15}\cdot\left(\frac{300^3}{3} - 0\right) = 4,800,000 \text{ ft}^3$$
This volume is one-eighth of the volume in Example 2.1.

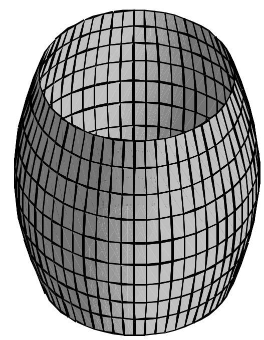
7. The key observation in this problem is that by simple proportions, had the steeple continued to a point it would have had height 36, hence 6 extra feet. One can copy the integration method, integrating only to 30, or one can subtract the volume of the missing "point" from the full pyramid. Either way the answer is
$$\frac{3^2 36}{3} - \left(\frac{1}{2}\right)^2 \cdot \frac{6}{3} = \frac{215}{2} \text{ ft}^3.$$

8. This volume is easily computed using elementary geometry formulas. Using calculus and the triangular cross sections, the area of cross sections is 150, so the total volume is
$$V = \int_0^{60} 150 dx = 9000.$$

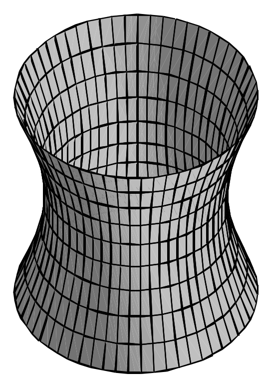
9. $V = \int_0^{60} \pi x^2 dy = \pi \int_0^{60} 60[60-y]dy$
$$= 60\pi\left[60y - \frac{y^2}{2}\right]_0^{60} = 60\pi\left[60^2 - \frac{60^2}{2}\right]$$
$$= \frac{60^3\pi}{2} = 108000\pi \text{ ft}^3$$

10. The radius of the cross-section is given by $r = x$, therefore the volume is given by
$$V = \int_0^{120} \pi x^2 dy = \pi \int_0^{120} 120\,(120-y)dy$$
$$= 120\pi \cdot \left[120y - \frac{y^2}{2}\right]_0^{120}$$
$$= 120\pi\left[120^2 - \frac{120^2}{2}\right]$$
$$= \frac{120^3\pi}{2} = 864,000\pi ft^3.$$

11. $V = \pi \int_0^{2\pi} \left(4 + \sin\frac{x}{2}\right)^2 dx$
$$= \pi \int_0^{2\pi} \left(16 + 8\sin\frac{x}{2} + \sin^2\frac{x}{2}\right) dx$$
$$= \pi\left(16x - 16\cos\frac{x}{2} + \frac{1}{2}x - \frac{1}{2}\sin x\right)\Big|_0^{2\pi}$$
$$= 33\pi^2 + 32\pi \text{ in}^3$$



12. $V = \int_0^{2\pi} \pi\left(4 - \sin\frac{x}{2}\right)^2 dx$
$$= \int_0^{2\pi} \pi\left(16 - 8\sin\frac{x}{2} + \sin^2\frac{x}{2}\right) dx$$
$$= 33\pi^2 - 32\pi \text{ in}^3$$



13. $V = \int_0^1 A(x)dx$
$$\approx \frac{1}{3(10)}[A(0) + 4A(.1) + 2A(.2)$$
$$+ 4A(.3) + 2A(.4) + 4A(.5)$$
$$+ 2A(.6) + 4A(.7) + 2A(.8)$$
$$+ 4A(.9) + A(1.0)]$$
$$= \frac{7.4}{30} \approx 0.2467\text{cm}^3$$

14. $V = \int_0^{1.2} A(x)dx$
$\approx \frac{0.2}{3}[f(0.0) + 4f(0.2) + 2f(0.4)$
$+ 4f(0.6) + 2f(0.8) + 4f(1.0)$
$+ f(1.2)]$
$= \frac{0.2}{3}[0 + 4(0.2) + 2(0.3) + 4(0.2)$
$+ 2(0.4) + 4(0.2) + 0]$
$\approx 0.253333.$

15. $V = \int_0^{2} A(x)dx$
$\approx \frac{2}{3(4)}[A(0) + 4A(.5) + 2A(1)$
$+4A(1.5) + A(2)]$
$= 2.5 \text{ ft}^3$

16. $V = \int_0^{0.8} A(x)dx$
$\approx \frac{0.1}{3}[f(0.0) + 4f(0.1) + 2f(0.2)$
$+ 4f(0.3) + 2f(0.4) + 4f(0.5)$
$+ 2f(0.6) + 4f(0.7) + f(0.8)]$
$= \frac{0.1}{3}[2.0 + 4(1.8) + 2(1.7) + 4(1.6)$
$+ 2(1.8) + 4(2.0) + 2(2.1) + 4(2.2)$
$+ 2.4]$
≈ 1.533333

17. (a) $V = \pi \int_0^2 (2-x)^2 dx$
$= -\pi \left(\frac{(2-x)^3}{3}\right)\Big|_0^2$
$= \frac{8\pi}{3}$

(b) $V = \pi \int_0^2 \left[3^2 - \{3 - (2-x)\}^2\right] dx$
$= \pi \int_0^2 \left[9 - \{1+x\}^2\right] dx$
$= \pi \left[9x|_0^2 - \frac{(1+x)^3}{3}\Big|_0^2 \right]$
$= \pi \left[18 - \frac{3^3 - 1^3}{3}\right] = \frac{28\pi}{3}$

18. (a) $V = \pi \int_{-\sqrt{2}}^{\sqrt{2}} \left[\left(4 - x^2\right)^2 - \left(x^2\right)^2\right] dx$
$= \pi \left[16x - \frac{8x^3}{3}\right]\Big|_{-\sqrt{2}}^{\sqrt{2}}$
$= \pi \left(\frac{64\sqrt{2}}{3}\right)$

(b) $V = \pi \int_{-\sqrt{2}}^{\sqrt{2}} \left(4 - x^2\right)^2 - \left(x^2\right)^2 dx$
$= \pi \left(\frac{64\sqrt{2}}{3}\right)$

19. (a) $V = \pi \int_0^2 (y^2)^2 dy = \pi \int_0^2 y^4 dy$
$= \pi \left(\frac{y^5}{5}\right)\Big|_0^2 = \frac{32\pi}{5}$

(b) $V = \pi \int_0^2 (4)^2 dy$
$- \pi \int_0^2 (4 - y^2)^2 dy$
$= \pi \int_0^2 (-y^4 + 8y^2) dy$
$= \pi \left(-\frac{y^5}{5} + \frac{8y^3}{3}\right)\Big|_0^2$
$= \pi \left[\left(-\frac{32}{5} + \frac{64}{3}\right) - (0+0)\right]$
$= \frac{224\pi}{15}$

20. (a) $V = \pi \int_0^1 (\sqrt{y})^2 dy - \pi \int_0^1 \left(y^2\right)^2 dy$
$= \pi \left(\frac{y^2}{2} - \frac{y^5}{5}\right)\Big|_0^1$
$= \pi \left(\frac{1}{2} - \frac{1}{5}\right)$
$= \frac{3\pi}{10}$

(b) $V = \pi \int_0^1 \left(1 - y^2\right)^2 dy - \pi \int_0^1 (1 - \sqrt{y})^2 dy$
$= \pi \int_0^1 \left(y^4 - 2y^2 - y + 2\sqrt{y}\right) dy$
$= \pi \left(\frac{y^5}{5} - \frac{2y^3}{3} - \frac{y^2}{2} + \frac{4y^{\frac{3}{2}}}{3}\right)\Big|_0^1$
$= \pi \left(\frac{1}{5} - \frac{2}{3} - \frac{1}{2} + \frac{4}{3}\right) = \frac{11\pi}{30}$

21. (a) $V = 4\pi e^2 - \pi \int_1^{e^2} (\ln y)^2 dy$
$= 4\pi e^2$
$- \left[y(\ln y)^2 - 2y \ln y + 2y\right]\Big|_1^{e^2}$
$= 4\pi e^2 - (2e^2 - 2)$
$= 2\pi(e^2 + 1).$

(b) $V = \pi \int_0^2 (e^x + 2)^2\, dx$

$- \pi \int_0^2 (2)^2 dx$

$= \pi \int_0^2 \left(e^{2x} + 4e^x\right) dx$

$= \pi \left(\frac{e^{2x}}{2} + 4e^x \right) \Big|_0^2$

$= \pi \left[\left(\frac{e^4}{2} + 4e^2 \right) - \left(\frac{1}{2} + 4 \right) \right]$

$= \pi \left(\frac{e^4}{2} + 4e^2 - \frac{9}{2} \right)$

22. (a) $V = \pi \int_{-\pi/4}^{\pi/4} [2^2 - (2 - \sec x)^2] dx$

$= \left(4\pi \int_{-\pi/4}^{\pi/4} \sec x dx \right)$

$= - \left(\pi \tan x|_{-\pi/4}^{\pi/4} \right)$

≈ 15.868

(b) $V = \pi \int_{-\pi/4}^{\pi/4} \sec^2 x dx$

$= \pi \tan x|_{-\pi/4}^{\pi/4} = 2\pi$

23. (a) $V = \pi \int_0^1 \left(\sqrt{\frac{x}{x^2 + 2}} \right)^2 dx$

$= \frac{\pi}{2} \ln |x^2 + 2||_0^1$

$= \frac{\pi}{2} \ln \frac{3}{2} \approx 0.637$

(b) $V = \pi \int_0^1 \left[3^2 - \left(3 - \sqrt{\frac{x}{x^2 + 2}} \right)^2 \right] dx$

$= \pi \int_0^1 \left(6\sqrt{\frac{x}{x^2 + 2}} - \frac{3x}{x^2 + 2} \right) dx$

$= 6\pi \int_0^1 \sqrt{\frac{x}{x^2 + 2}} dx$

$= - \frac{3\pi}{2} \ln |x^2 + 2| \Big|_0^1$

≈ 7.4721

24. $e^{-x^2} = x^2$ when $x \approx \pm 0.753$

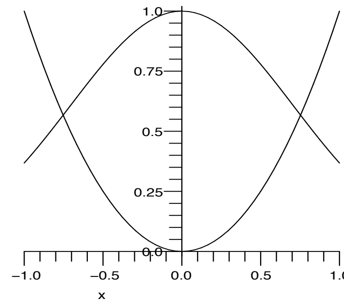
(a) $V = \pi \int_{0.753}^{0.753} [(e^{-x^2})^2 - (x^2)^2] dx$

≈ 3.113

(b) $V = \pi \int_{0.753}^{0.753} [(e^{-x^2} + 1)^2$

$- (x^2 + 1)^2] dx$

≈ 9.266



25. (a) $V = \int_0^4 \pi \left(\frac{4 - y}{2} \right)^2 dy$

$= \frac{\pi}{4} \int_0^4 \left(16 - 8y + y^2\right) dy$

$= \frac{\pi}{4} \left[16y - 4y^2 + \frac{y^3}{3} \right]_0^4$

$= \frac{\pi}{4} \left[64 - 64 + \frac{64}{3} \right] = \frac{16\pi}{3}$

(b) $V = \int_0^2 \pi \left(4 - 2x\right)^2 dx$

$= \pi \int_0^2 \left(16 - 16x + 4x^2\right) dx$

$= \pi \left[16x - 16\frac{x^2}{2} + \frac{4x^3}{3} \right]_0^2$

$= \pi \left[32 - 32 + \frac{32}{3} \right] = \frac{32\pi}{3}$

(c) $V = \int_0^2 \pi (4)^2 dx - \int_0^2 \pi (2x)^2 dx$

$= \pi \int_0^2 \left(16 - 4x^2\right) dx$

$= \pi \left[16x - \frac{4x^3}{3} \right]_0^2$

$= \pi \left[32 - \frac{32}{3} \right] = \frac{64\pi}{3}$

(d) $V = \int_0^2 \pi (8 - 2x)^2 dx - \int_0^2 \pi (4)^2 dx$

$= \pi \int_0^2 \left(64 - 32x + 4x^2 - 16\right) dx$

$= \pi \left[48x - 32\frac{x^2}{2} + \frac{4x^3}{3} \right]_0^2$

$$= \pi\left[96 - 64 + \frac{32}{3}\right] = \frac{128\pi}{3}$$

(e) $$V = \int_0^4 \pi(2)^2 dy - \int_0^4 \pi\left(\frac{y}{2}\right)^2 dy$$
$$= \pi\int_0^4 \left(4 - \frac{y^2}{4}\right) dy$$
$$= \pi\left[4y - \frac{1}{4}\cdot\frac{y^3}{3}\right]_0^4$$
$$= \pi\left[16 - \frac{16}{3}\right] = \frac{32\pi}{3}$$

(f) $$V = \int_0^4 \pi\left(\frac{8-y}{2}\right)^2 dy - \int_0^4 \pi(2)^2 dy$$
$$= \pi\int_0^4 \left(\frac{64 - 16y + y^2}{4} - 4\right) dy$$
$$= \frac{\pi}{4}\left[64y - 16\frac{y^2}{2} + \frac{y^3}{3} - 16y\right]_0^4$$
$$= \pi\left[64 + \frac{64}{3}\right] = \frac{256\pi}{3}$$

26. (a) $$V = \int_{-2}^{2} \pi\left(4 - x^2\right)^2 dx = \frac{512\pi}{15}$$

(b) $$V = \int_0^4 \pi(\sqrt{y})^2 dy = 8\pi$$

(c) $$V = \int_{-2}^{2} \pi\left[(6 - x^2)^2 - 2^2\right] dx$$
$$= \frac{384\pi}{5}$$

(d) $$V = \int_{-2}^{2} \pi\left[6^2 - (2 + x^2)^2\right] dx$$
$$= \frac{1408\pi}{15}$$

(e) $$V = \int_0^4 \pi\left[(2 + \sqrt{y})^2 - (2 - \sqrt{y})^2\right] dy$$
$$= \int_0^4 8\pi\, y^{1/2} dy = \frac{16}{3}\pi y^{3/2}\Big|_0^4$$
$$= \frac{128}{3}\pi$$

(f) $$V = \int_0^4 \pi\left[(4 + \sqrt{y})^2 - (4 - \sqrt{y})^2\right] dy$$
$$= \int_0^4 16\pi\, y^{1/2} dy = \frac{32}{3}\pi y^{3/2}\Big|_0^4$$
$$= \frac{256}{3}\pi$$

27. (a) $$V = \int_0^1 \pi(1)^2 dy - \int_0^1 \pi\left(\sqrt{y}\right)^2 dy$$
$$= \pi\int_0^1 (1 - y) dy$$
$$= \pi\left(y - \frac{y^2}{2}\right)\Bigg|_0^1 = \frac{\pi}{2}$$

(b) $$V = \int_0^1 \pi\left(x^2\right)^2 dx$$
$$= \pi\frac{x^5}{5}\Bigg|_0^1 = \frac{\pi}{5}$$

(c) $$V = \int_0^1 \pi\left(1 - \sqrt{y}\right)^2 dy$$
$$= \pi\int_0^1 \left(1 - 2y^{1/2} + y\right) dy$$
$$= \pi\left(y - \frac{4}{3}y^{3/2} + \frac{y^2}{2}\right)\Bigg|_0^1 = \frac{\pi}{6}$$

(d) $$V = \int_0^1 \pi(1)^2 dx - \int_0^1 \pi\left(1 - x^2\right)^2 dx$$
$$= \pi\int_0^1 \left(2x^2 - x^4\right) dx$$
$$= \pi\left(\frac{2}{3}x^3 - \frac{x^5}{5}\right)\Bigg|_0^1 = \frac{7\pi}{15}$$

(e) $$V = \int_0^1 \pi(2)^2 dy - \int \pi\left(1 + \sqrt{y}\right)^2 dy$$
$$= \pi\int_0^1 \left(3 - 2y^{1/2} - y\right) dy$$
$$= \pi\left(3y - \frac{4}{3}y^{3/2} - \frac{y^2}{2}\right)\Bigg|_0^1 = \frac{7\pi}{6}$$

(f) $$V = \int_0^1 \pi\left(x^2 + 1\right)^2 dx$$
$$= -\int_0^1 \pi(1)^2 dx$$
$$= \pi\int_0^1 \left(x^4 + 2x^2\right) dx$$
$$= \pi\left(\frac{x^5}{5} + \frac{2}{3}x^3\right)\Bigg|_0^1 = \frac{13\pi}{15}$$

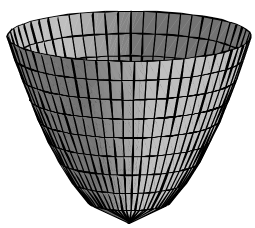
28. (a) $$V = \int_0^1 \pi x^2 dx = \frac{\pi}{3}$$

(b) $$V = \int_{-1}^{0} \pi\left[1 - (1 + y)^2\right] dy$$
$$+ \int_0^1 \pi\left[1 - (1 - y)^2\right] dy$$
$$= \frac{2\pi}{3} + \frac{2\pi}{3} = \frac{4\pi}{3}$$

(c) $$V = \int_0^1 \pi\left[(1 + x)^2 - (1 - x)^2\right] dx$$
$$= 2\pi$$

(d) $V = \int_0^1 \pi \left[(1+x)^2 - (1-x)^2\right] dx$
$= 2\pi$

29. $V = \pi \int_0^h \left(\sqrt{\frac{y}{a}}\right)^2 dy$
$= \frac{\pi}{a} \int_0^h y dy = \frac{\pi h^2}{2a}$
The volume of a cylinder of height h and radius $\sqrt{h/a}$ is $h \cdot \pi(\sqrt{h/a})^2 = \frac{\pi h^2}{a}$



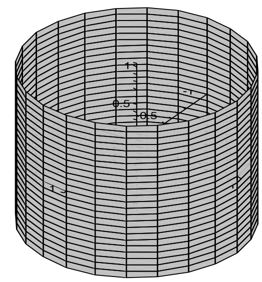
30. The confusing thing here is that the h of Exercise 29 is not the h of this problem. Realizing this,
$V = \frac{\pi(h/a)^2}{2a} = \frac{\pi h^2}{2a^3}$

31. We can choose either x or y to be our integration variable,
$V = \pi \int_{-1}^1 dx = \pi x|_{-1}^1 = 2\pi$

32. This is, of course, a solid ball. Notice that $y = \sqrt{1-x^2}$.
$V = \int_{-1}^1 \pi(\sqrt{1-x^2})^2 dx = \frac{4\pi}{3}$

33. The line connecting the two points $(0,1)$ and $(1,-1)$ has equation
$y = -2x+1$ or $x = \frac{1-y}{2}$.
$V = \int_{-1}^1 \pi \left(\frac{1-y}{2}\right)^2 dy$
$= \pi \left(\frac{y}{4} - \frac{y^2}{4} + \frac{y^3}{12}\right)\Big|_{-1}^1 = \frac{2\pi}{3}$

34. The fact that the ratios is $3:2:1$ is easy to confirm since we know the volumes are 2π, $\frac{4\pi}{3}$ and $\frac{2\pi}{3}$.



35. $V = \pi \int_{-r}^r \left(\sqrt{r^2-y^2}\right)^2 dy$
$= \pi \int_{-r}^r (r^2-y^2) dy$
$= \pi \left(r^2 y - \frac{y^3}{3}\right)\Big|_{-r}^r = \frac{4}{3}\pi r^3$

36. $V = \int_0^h \pi \left(-\frac{r}{h}y + r\right)^2 dy = \frac{\pi r^2 h}{3}$

37. If we compute the two volumes using disks parallel to the base, we have identical cross sections, so the volumes are the same.

38. They have the same areas. This can be seen by using elementary geometrical formulas for area or by considering integrals. The area of the parallelograms is given by the integral of the heights of the line segments from 0 to 5. The heights of the line segments are equal.

39. (a) If each of these line segments is the base of square, then the cross-sectional area is evidently
$A(x) = 4(1-x^2)$.
The volume would be
$V_a = \int_{-1}^1 A(x) dx$
$= 2\int_0^1 A(x) dx = 8\left(x - \frac{x^3}{3}\right)\Big|_0^1 = \frac{16}{3}$.

(b) These segments I_x cannot be the literal "bases" of circles, because circles "sit" on a single point of tangency. They could however be diameters. Assuming so, the cross sectional area would be "$\pi/2$ times radius-squared" or $\pi(1-x^2)/2$. The resulting volume would be $\pi/8$ times the previous case, or $2\pi/3$.

40. (a) $V = \int_{-1}^0 [2(x+1)]^2 dx = \frac{4}{3}$

(b) Note that the area of an equilateral triangle with side length l is $\sqrt{3}l^2/4$. This means that for a slice we have
$A(x) = \sqrt{3}(x+1)^2/4$
and
$$V = \int_{-1}^{0} \frac{\sqrt{3}(x+1)^2}{4}dx = \frac{\sqrt{3}}{12}$$

41. Reasoning as in Exercise 39, the line segment I_x is $[x^2, 2-x^2], (1 \le x \le 1)$. The length of this segment is
$(2-x^2) - x^2 = 2(1-x^2)$,
hence in case (a)
$A(x) = 4(1-x^2)^2 = 4(1-2x^2+x^4)$.
The volume would again be
$$V = 2\int_0^1 A(x)dx$$
$$= 8\left(x - \frac{2x^3}{3} + \frac{x^5}{5}\right)\Big|_0^1$$
$$= 8\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{64}{15}.$$
With the same provisos as in Exercise 39, the answer to (b) would be $\pi/8$ times the (a)-case, or $8\pi/15$.

For (c), the volume would be $\sqrt{3}/4$ times the (a)-case, or $16\sqrt{3}/15$.

42. (a) In this case, $A(x) = (\ln x)^2$ and
$$V = \int_1^2 (\ln x)^2 dx$$
$$= 2(\ln 2)^2 - 4\ln 2 + 2.$$
(b) In this case, $A(x) = \dfrac{\pi}{2}\left(\dfrac{\ln x}{2}\right)^2$ and
$$V = \int_1^2 \frac{\pi}{2}\left(\frac{\ln x}{2}\right)^2 dx$$
$$= \frac{(\ln 2)^2}{4} - \frac{\ln 2}{2} + \frac{1}{4}.$$

43. This time the line segment I_x is $[0, e^{-2x}]$, $(0 \le x \le \ln 5)$. If (a) this is the base of a square, the cross-sectional area is $A(x) = (e^{-2x})^2 = e^{-4x}$. The volume V_a would be the integral
$$\int_0^{\ln 5} A(x)dx$$
$$= \int_0^{\ln 5} e^{-4x}dx = \frac{-e^{-4x}}{4}\Big|_0^{\ln 5}$$
$$= \frac{1-\left(\frac{1}{5}\right)^4}{4} = \frac{156}{625} = .2496.$$
In the (b)-case, the segment I_x is the base of a *semi*circle, so the cross-sectional area would be
$$\left(\frac{1}{2}\right)\pi\left(\frac{e^{-2x}}{2}\right)^2 = \left(\frac{\pi}{8}\right)e^{-4x}.$$
The resulting volume V_b would be
$(\pi/8)V_a = \dfrac{39\pi}{1250} \approx .09802.$

44. (a) In this case, $A(x) = (x^2 - \sqrt{x})^2$ and
$$V = \int_0^1 (x^2 - \sqrt{x})^2 dx = \frac{9}{70}$$
(b) In this case,
$$A(x) = \pi\left(\frac{x^2-\sqrt{x}}{2}\right)^2 \text{ and}$$
$$V = \int_0^1 \pi\left(\frac{x^2-\sqrt{x}}{2}\right)^2 dx = \frac{9}{280}$$

45. We must estimate $\pi\int_0^3 [f(x)]^2 dx$.

The given table can be extended to give these respective values for

$f(x)2: 4, 1.44, .81, .16, 1.0, 1.96, 2.56.$

Simpson's approximation to the integral would be
$$\frac{3}{(3)(6)}\{4 + 4(1.44) + 2(.81)$$
$$+ 4(.16) + 2(1.0) + 4(1.96) + 2.56\}.$$
The sum in the braces is 24.42, and this must be multiplied by $\pi/6$ giving a final answer of 12.786.

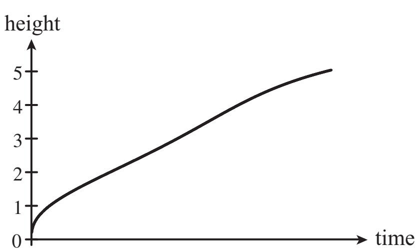
46. Use Simpson's rule.
$$V = \int_0^2 \pi[f(x)]^2 dx$$
$$\approx \frac{\pi(0.25)}{3}[(4.0)^2 + 4(3.6)^2 + 2(3.4)^2$$
$$+ 4(3.2)^2 + 2(3.5)^2 + 4(3.8)^2 + 2(4.2)^2$$
$$+ 4(4.6)^2 + (5.0)^2]$$
$$\approx 94.01216$$

47. In this problem, let $x = g(y)$ be the equation of the given curve describing the shape of the container. For each height y, let $V(y)$ be the volume of fluid in the container when the depth is y. Later we will estimate $V(y)$. For now, one knows that $V(y)$ is the integral of $\pi[g(y)]^2$, or by the fundamental theorem of calculus, that
$$\frac{dV}{dy} = \pi[g(y)]^2.$$
In actual practice, y and hence V are functions of t (time). Our primary interest is in y as a function of t, but we will obtain this information indirectly, first finding V as a function of y. It appears that $g(y)$ is about $2y$ for $0 < y < 1$,

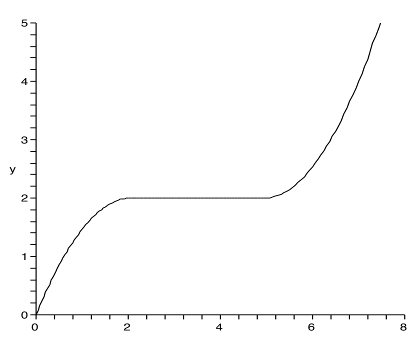
which leads to $[g(y)]^2 = 4y^2$, $V(y) = 4\pi y^3/3$ (on $0 < y < 1$), and $V(1) = 4\pi/3 = 4.2$. We'll keep the *formula* in mind for later, but for now will use the value at $y = 1$ and the crude trapezoidal estimate
$V(y+1) = V(y) + \pi[g^2(y) + g^2(y+1)]/2$
to compile the following table:

y	$g(y)$	$g^2(y)$	$V(y)$
1	2	4	4.2
2	2	9	24.6
3	3	9	52.9
4	3	9	81.2
5	4	16	120.4

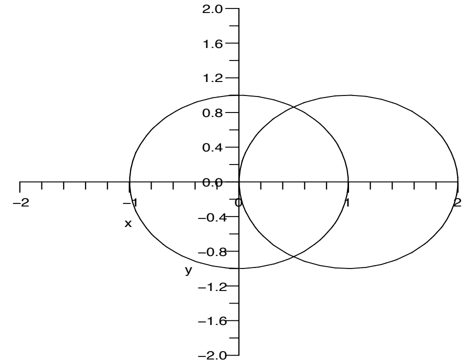
The assumption of uniform flow rate amounts to $dV/dt =$ constant, and if we start the clock ($t = 0$) as we begin the flow, we get $V = kt$ for some k. The above table, supplemented by the formula when $y < 1$, can be read to give y (vertical) as a function of V (horizontal). But because $V = kt$, the graph looks exactly the same if the horizontal units are time. In the following picture, we have scaled it on the assumption of a flow rate of 120.4 cubic units per minute, a rate which requires one minute to fill the container. The previous formula $4\pi y^3/3 = V(= kt = (120.4)t)$ (on $0 < y < 1$), becomes $y = (3.06)t^{1/3}$ for very small t, and accounts for the (barely discernible) vertical tangent at $t = 0$.



48.



49.



For the points of intersection, solve
$1 - (x-1)^2 = 1 - x^2$
thatis,$x^2 - 2x + 1 = x^2$
or $x = \dfrac{1}{2} \Rightarrow y = \pm\dfrac{\sqrt{3}}{2}$
The desired volume V is the sum of the volume V_1 generated by revolving the arc of the circle $x^2 + y^2 = 1$ about the x-axis from $x = \frac{1}{2}$ to $x = 1$ and the volumeV_2 generated by revolving the arc of the circle $(x-1)^2 + y^2 = 1$ about the x-axis from $x = 0$ to $x = \frac{1}{2}$.

Therefore $V = V_1 + V_2$ where,

$$V_1 = \pi \int_{1/2}^{1} (1 - x^2)dx = \pi \left(x - \frac{x^3}{3}\right)\Bigg|_{1/2}^{1}$$

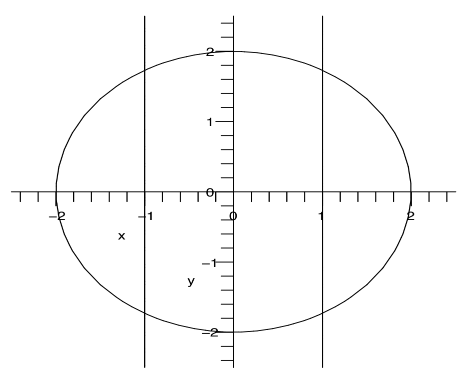
$$= \pi\left[\left(1 - \frac{1}{3}\right) - \left(\frac{1}{2} - \frac{1}{24}\right)\right] = \frac{5\pi}{24}$$

and $V_2 = \pi \displaystyle\int_0^{1/2} \left(1 - (x-1)^2\right)dx$

$$= \pi \int_0^{1/2} (2x - x^2)dx = \pi\left[x^2 - \frac{x^3}{3}\right]\Bigg|_0^{1/2}$$

$= \dfrac{5\pi}{24}$
$V = V_1 + V_2 \approx 1.308997$

50.



The required region is formed by intersection of revolving circle $x^2 + y^2 = 4$ about y-axis and revolving $x = 1, -4 \le y \le 4$about y-axis. Desired volume V is the volume obtained by revolving the shaded region R about the x-axis where R is bounded by $x = 0, x = 1$ and the arc of the circle $x^2 + y^2 = 4$

$x = 1 \Rightarrow y = \pm\sqrt{3}$
$R = R_1 + R_2 + R_3$
R1 is bounded by $x = 0, x^2 + y^2 = 4, y = \sqrt{3}$
R2 is bounded by $x = 0, y = \sqrt{3}, y = -\sqrt{3}$
R3 is bounded by $x = 0, x^2 + y^2 = 4, y = -\sqrt{3}$
Let V1 ,V2 ,V3 be the respective volumes obtained by revolving R1 , R2 , R3 about y-axis

$$V_1 = \int_{\sqrt{3}}^{2} \pi\left(4 - y^2\right)dy$$

$$= \pi\left[4y - \frac{y^3}{3}\right]\Bigg|_{\sqrt{3}}^{2} = \pi\left(\frac{16}{3} - \frac{8\sqrt{3}}{3}\right)$$

$$V_2 = \pi\int_{-\sqrt{3}}^{\sqrt{3}} 1dy = 2\pi\sqrt{3}$$

$V_3 = V_1$
$V = V_1 + V_2 + V_3$

$$= \frac{2\pi}{3}\left(16 - 5\sqrt{3}\right)$$

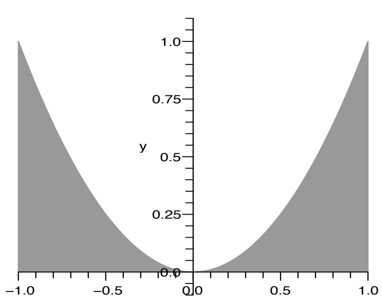
5.3 Volumes by Cylindrical Shells

1. Radius of a shell: $r = 2 - x$
Height of a shell: $h = x^2$

$$V = \int_{-1}^{1} 2\pi(2 - x)x^2dx$$

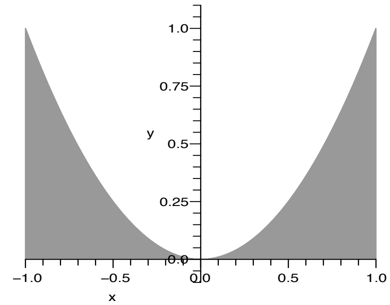
$$= 2\pi\left(\frac{2x^3}{3} - \frac{x^4}{4}\right)\Bigg|_{-1}^{1}$$

$$= \frac{8\pi}{3}$$



2. Radius of a shell: $r = 2 + x$
Height of a shell: $h = x^2$

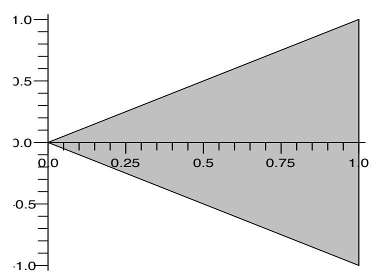
$$V = \int_{-1}^{1} 2\pi(2 + x)x^2dx = \frac{8\pi}{3}$$



3. Radius of a shell: $r = x$
Height of a shell: $h = 2x$

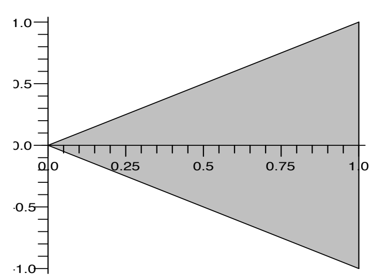
$$V = \int_{0}^{1} 2\pi x(2x)dx$$

$$= \frac{4\pi}{3}x^3\Big|_{0}^{1} = \frac{4\pi}{3}$$



4. Radius of a shell: $r = 2 - x$.
Height of a shell: $h = 2x$.

$$V = \int_{0}^{1} 2\pi(2 - x)(2x)dx = \frac{8\pi}{3}$$



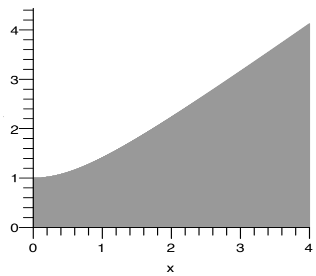
5. Radius of a shell: $r = x$.
eight of a shell: $h = f(x) = \sqrt{x^2 + 1}$.

$$V = \int_{0}^{4} 2\pi x\sqrt{x^2 + 1}dx$$

$$= \pi \int_0^4 2x\sqrt{x^2+1}\,dx$$

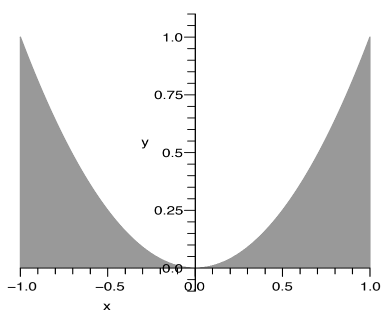
$$= \pi \left(\frac{2\left(x^2+1\right)^{\frac{3}{2}}}{3} \right)\Bigg|_0^4 = \frac{2\pi}{3}\left[(17)^{\frac{3}{2}} - 1 \right]$$

$$\approx 144.7076$$



6. Radius of a shell: $r = 2 - x$.
Height of a shell: $h = f(x) = x^2$.

$$V = \int_{-1}^{1} 2\pi\,(2-x)\,x^2 dx = \tfrac{8\pi}{3}$$



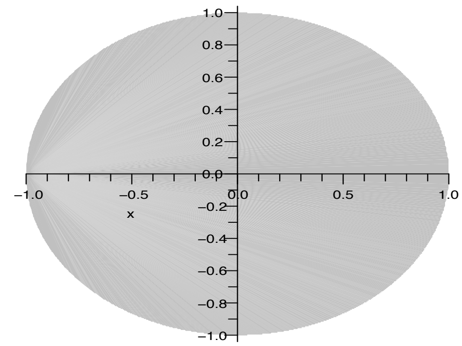
7. Radius of a shell: $r = 2 - y$.
Height of a shell: $h = f(y) = 2\sqrt{1-y^2}$.

$$V = \int_{-1}^{1} 2\pi\,(2-y)\,2\sqrt{1-y^2}dy$$

$$= 4\pi \int_{-1}^{1} (2-y)\sqrt{1-y^2}dy$$

$$= 8\pi \int_{-1}^{1} \sqrt{1-y^2}dy - 4\pi \int_{-1}^{1} y\sqrt{1-y^2}dy$$

$$= 16\pi \left(\frac{\pi}{4}\right) - 0 = 4\pi^2$$



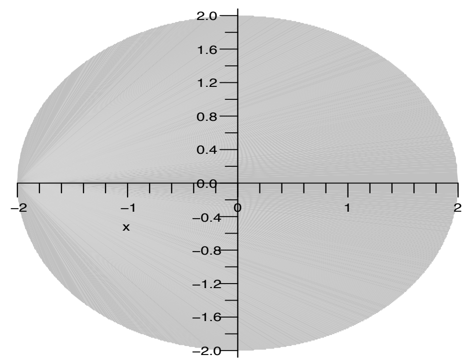
8. Radius of a shell: $r = 4 - y$.
Height of a shell: $h = f(y) = 2\sqrt{4-y^2}$.

$$V = \int_{-2}^{2} 2\pi\,(4-y)\,2\sqrt{4-y^2}dy$$

$$= 4\pi \int_{-2}^{2} (4-y)\sqrt{4-y^2}dy$$

$$= 2\left(8\pi \int_{-2}^{2} \sqrt{4-y^2}dy - 2\pi \int_{-2}^{2} y\sqrt{4-y^2}dy \right)$$

$$= 2\left(8\pi\,(2\pi)\right) - 0 = 32\pi^2$$



9. $$V = \int_{-1}^{1} 2\pi(x+2)\left((2-x^2)-x^2\right)dx$$

$$= 2\pi \int_{-1}^{1} \left(4 + 2x - 4x^2 - 2x^3\right)dx$$

$$= 2\pi \left(4x + x^2 - \frac{4x^3}{3} - \frac{x^4}{2} \right)\Bigg|_{-1}^{1}$$

$$= \frac{32\pi}{3}$$

10. $$V = \int_{-1}^{1} 2\pi(2-x)\left((2-x^2)-x^2\right)dx$$

$$= 2\pi \int_{-1}^{1} \left(4 - 2x - 4x^2 + 2x^3\right)dx$$

$$= 2\pi \left(4x - x^2 - \frac{4x^3}{3} + \frac{x^4}{2} \right)\Bigg|_{-1}^{1}$$

$$= \frac{32\pi}{3}$$

11. $V = \int_{-2}^{2} 2\pi(2+y)(4-y^2)dy$
$= 2\pi\left(8y + 2y^2 - \frac{2y^3}{3} - \frac{y^4}{4}\right)\Big|_{-2}^{2}$
$= \frac{128\pi}{3}$

12. $V = \int_{-2}^{2} 2\pi(2-y)(4-y^2)dy$
$= 2\pi\left(8y - 2y^2 - \frac{2y^3}{3} + \frac{y^4}{4}\right)\Big|_{-2}^{2}$
$= \frac{128\pi}{3}$

13. $V = \int_{0}^{2} 2\pi\,(3-x)\,(e^x - x - 1)\,dx$
$= 2\pi \int_{0}^{2} \big((3-x)\,e^x - 2x + x^2 - 3\big)dx$
$= 2\pi\left[[(4-x)\,e^x - x^2 + \frac{x^3}{3} - 3x]\right]\Big|_{0}^{2}$
$= 2\pi\left[\left(2e^2 - 4 + \frac{8}{3} - 6\right) - (4-3)\right]$
≈ 21.6448

14. $V = \int_{-1}^{2} 2\pi(3-x)(x-(x^2-2))dx$
$= 2\pi \int_{-1}^{2} \big(6 + x - 4x^2 + x^3\big)dx$
$= 2\pi\left(6x + \frac{x^2}{2} - \frac{4x^3}{3} + \frac{x^4}{4}\right)\Big|_{-1}^{2}$
$= \frac{45\pi}{2}$

15. $V = \int_{-2}^{4} 2\pi(5-y)[9-(y-1)^2]dy$
$= \int_{-2}^{4} (y^3 - 7y^2 + 2y + 40)\ dy$
$= \left(\frac{y^4}{4} - \frac{7y^3}{3} + y^2 + 40y\right)\Big|_{-2}^{4}$
$= 288\pi$

16. $V = \int_{-2}^{4} 2\pi(3+y)[9-(y-1)^2]dy$
$= \int_{-2}^{4} (-y^3 - y^2 + 14y + 24)dy$
$= \left(-\frac{y^4}{4} - \frac{y^3}{3} + 7y^2 + 24y\right)\Big|_{-2}^{4}$
$= 288\pi$

17. (a) $V = \int_{2}^{4} 2\pi(y)\,(y-(4-y))\,dy$
$= 2\pi \int_{2}^{4} \big(2y^2 - 4y\big)dy$
$= 2\pi\left(\frac{2y^3}{3} - 2y^2\right)\Big|_{2}^{4}$
$= \frac{80\pi}{3}$

(b) $V = \int_{0}^{2} 2\pi(x)\,(4-(4-x))\,dx$
$= \int_{2}^{4} 2\pi(x)(4-x)dx$
$= 2\pi\left(\frac{x^3}{3}\right)\Big|_{0}^{2} + 2\pi\left(2x^2 - \frac{x^3}{3}\right)\Big|_{2}^{4}$
$= 2\pi\left(\frac{8}{3} + \frac{16}{3}\right) = 16\pi$

(c) $V = \int_{2}^{4} \pi\,(4-(4-y))^2\,dy$
$= \int_{2}^{4} \pi(4-y)^2 dy$
$= \pi \int_{2}^{4} y^2 dy$
$- \pi \int_{2}^{4} (16 - 8y + y^2)dy$
$= \pi \int_{2}^{4} (-16 + 8y)dy$
$= \pi\,(-16y + 4y^2)\big|_{2}^{4} = 16\pi$

(d) $V = \int_{2}^{4} 2\pi(4-y)\,(y-(4-y))\,dy$
$= 2\pi \int_{2}^{4} \big(-2y^2 + 12y - 16\big)dy$
$= 2\pi\left(-\frac{2y^3}{3} + 6y^2 - 16y\right)\Big|_{2}^{4}$
$= \frac{16\pi}{3}$

18. (a) $V = \pi \int_{-2}^{0} \left[(x+4)^2 - (-x)^2\right] dx$
$= \pi \int_{-2}^{0} (8x+16)dx$
$= \pi\,(4x^2 + 16x)\big|_{-2}^{0}$
$= 32\pi$

(b) $V = 2\pi \int_{-2}^{0} (2+x)\cdot[(x+2) - (-x-2)]dx$
$= 2\pi \int_{-2}^{0} (2x^2 + 8x + 8)dx$
$= 2\pi\left(\frac{2x^3}{3} + 4x^2 + 8x\right)\Big|_{-2}^{0}$
$= \frac{32\pi}{3}$

(c) $V = 2\pi \int_{-2}^{0} (-x) \cdot [(x+2) - (-x-2)] dx$

$= 2\pi \int_{-2}^{0} (-2x^2 - 4x) dx$

$= 2\pi \left(-\frac{2x^3}{3} - 2x^2 \right) \Big|_{-2}^{0}$

$= \frac{16\pi}{3}$

(d) $V = \pi \int_{-2}^{0} (x+2)^2 dx$

$= \pi \int_{-2}^{0} (x^2 + 4x + 4) dx$

$= \pi \left(\frac{x^3}{3} + 2x^2 + 4x\right) \Big|_{-2}^{0}$

$= \frac{8\pi}{3}$

19. (a) Method of shells.

$V = \int_{-2}^{3} 2\pi(3-x)[x - (x^2 - 6)] dx$

$= \int_{-2}^{3} 2\pi(-x^3 - 4x^2 - 3x + 18) dx$

$= \frac{625\pi}{6}$

(b) Method of washers.

$V = \int_{-2}^{3} \pi[(x^2-6)^2 - x^2] dx$

$= \int_{-2}^{3} \pi(x^4 - 13x^2 + 36) dx$

$= \frac{250\pi}{3}$

(c) Method of shells.

$V = \int_{-2}^{3} 2\pi(3+x)[x - (x^2 - 6)] dx$

$= \int_{-2}^{3} 2\pi(x^3 - 2x^2 + 9x + 18) dx$

$= \frac{875\pi}{6}$

(d) Method of washers.

$V = \int_{-2}^{3} \pi[(6+x)^2 - (x^2)^2] dx$

$= \int_{-2}^{3} \pi(-x^4 + x^2 + 12x + 36) dx$

$= \frac{500\pi}{3}$

20. (a) $V = \pi \int_{-1}^{2} [(3+y)^2 - (y^2+1)^2] dy$

$= \pi \int_{-1}^{2} (-y^4 - y^2 + 6y + 8) dy$

$= \pi \left(-\frac{y^5}{5} - \frac{y^3}{3} + 3y^2 + 8y \right) \Big|_{-1}^{2}$

$= \frac{117\pi}{5}$

(b) $V = 2\pi \int_{-1}^{2} (y+1)[((2+y) - y^2] dy$

$= 2\pi \int_{-1}^{2} (-y^3 + 3y + 2) dy$

$= 2\pi \left(-\frac{y^4}{4} + \frac{3y^2}{2} + 2y \right) \Big|_{-1}^{2}$

$= \frac{27\pi}{2}$

(c) $V = \pi \int_{-1}^{2} [(4+y)^2 - (y^2+2)^2] dy$

$= \pi \int_{-1}^{2} (-y^4 - 3y^2 + 8y + 12) dy$

$= \pi \left(-\frac{y^5}{5} - y^3 + 4y^2 + 12y \right) \Big|_{-1}^{2}$

$= \frac{162\pi}{5}$

(d) $V = 2\pi \int_{-1}^{2} (y+2)[(2+y) - y^2] dy$

$= 2\pi \int_{-1}^{2} (-y^3 - y^2 + 4y + 4)\, dy$

$= 2\pi \left(-\frac{y^4}{4} - \frac{y^3}{3} + 2y^2 + 4y \right) \Big|_{-1}^{2}$

$= \frac{45\pi}{2}$

21. (a) $V = \int_{0}^{1} \pi(2-x)^2 dx$

$- \int_{0}^{1} \pi \left(x^2\right)^2 dx$

$= \pi \int_{0}^{1} (x^2 - 4x + 4) dx$

$- \pi \int_{0}^{1} x^4 dx$

$= \pi \int_{0}^{1} (-x^4 + x^2 - 4x + 4) dx$

$= \pi \left(\frac{x^5}{5} + \frac{x^3}{3} - 2x^2 + 4x \right) \Big|_{0}^{1}$

$= \frac{32\pi}{15}$

(b) $V = \int_{0}^{1} 2\pi x \left(2 - x - x^2\right) dx$

$= 2\pi \int_{0}^{1} \left(2x - x^2 - x^3\right) dx$

$= 2\pi \left(x^2 - \frac{x^3}{3} - \frac{x^4}{4} \right) \Big|_{0}^{1}$

$$= \frac{5\pi}{6}$$

(c) $V = \int_0^1 2\pi(1-x)(2-x-x^2)dx$

$$= 2\pi \int_0^1 \left(x^3 - 3x + 2\right)dx$$

$$= 2\pi \left(\frac{x^4}{4} - \frac{3x^2}{2} + 2x\right)\Big|_0^1$$

$$= \frac{3\pi}{2}$$

(d) $V = \int_0^1 \pi \left(2 - 2x^2\right)^2 dx$

$$= \int_0^1 \pi \left(2 - (2-x)\right)^2 dx$$

$$= \pi \int_0^1 (x^4 - 4x^2 + 4)dx - \pi \int_0^1 x^2 dx$$

$$= \pi \int_0^1 (x^4 - 5x^2 + 4)dx$$

$$= \pi \left(\frac{x^5}{5} - \frac{5x^3}{3} + 4x\right)\Big|_0^1$$

$$= \frac{38\pi}{15}$$

22. (a) $V = \pi \int_0^1 [(2-x^2)^2 - x^2]dx$

$$= \pi \int_0^1 (x^4 - 5x^2 + 4)dx$$

$$= \pi \left(\frac{x^5}{5} - \frac{5x^3}{3} + 4x\right) dx$$

$$= \frac{97\pi}{60}$$

(b) $V = 2\pi \int_0^1 x(2-x^2-x)dx$

$$= 2\pi \int_0^1 (-x^3 - x^2 + 2x)dx$$

$$= 2\pi \left(-\frac{x^4}{4} - \frac{x^3}{3} + x^2\right) dx$$

$$= \frac{3\pi}{5}$$

(c) $V = 2\pi \int_0^1 (x+1)(2-x^2-x)dx$

$$= 2\pi \int_0^1 (-x^3 - 2x^2 + x + 2)dx$$

$$= 2\pi \left(-\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} + 2x\right) dx$$

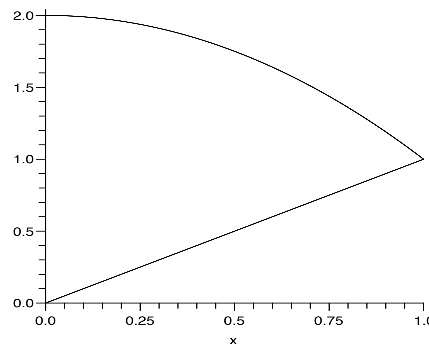
$$= \frac{21\pi}{10}$$

(d) $V = \pi \int_0^1 [(2-x^2+1)^2$
$- (x+1)^2]dx$

$$= \pi \int_0^1 (x^4 - 7x^2 - 2x + 8)dx$$

$$= \pi \left(\frac{x^5}{5} - \frac{7x^3}{3} - x^2 + 8x\right) dx$$

$$= \frac{187\pi}{60}$$



23. (a) $V = 2\pi \int_0^1 y(2-y-y^2)dy$

$$= 2\pi \int_0^1 (-y^3 - y^2 + 2y)dy$$

$$= 2\pi \left(-\frac{y^4}{4} - y^3 3 + y^2\right)\Big|_0^1$$

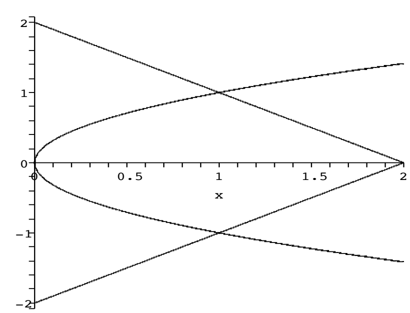
$$= \frac{5\pi}{6}$$

(b) $V = 2\pi \int_0^1 [(2-y)^2 - (y^2)^2]dy$

$$= 2\pi \int_0^1 (-y^4 + y^2 - 4y + 4)dy$$

$$= 2\pi \left(-\frac{y^5}{5} + \frac{y^3}{3} - 2y^2 + 4y\right)\Big|_0^1$$

$$= \frac{64\pi}{15}$$

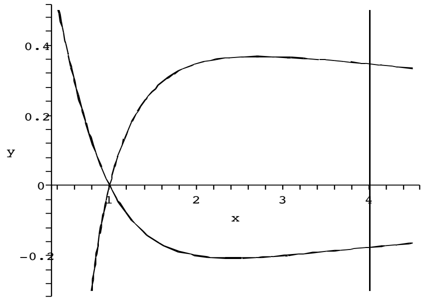


24. (a) $V \approx 2\pi \int_0^{0.79} y[(2-y) - \ln(y+1)]dy$

$$\approx 2.08$$

(b) $V \approx \pi \int_0^{0.79} [(2-y)^2 - \ln^2(y+1)]dy$

≈ 6.20

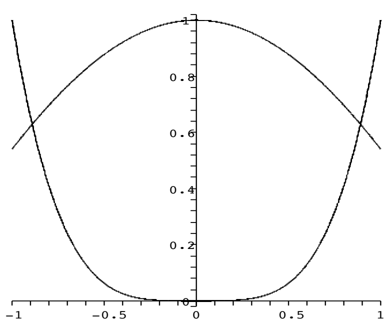


25. (a) $V \approx 2\pi \int_{-0.89}^{0.89} (2-x)\cdot(\cos x - x^4)\, dx$
≈ 16.72

(b) $V \approx \pi \int_{-0.89}^{0.89} [(2-x^4)^2 - (2-\cos x)^2] dx$
≈ 12.64

(c) $V \approx \pi \int_{-0.89}^{0.89} [(\cos x)^2 - (x^4)^2] dx$
≈ 4.09

(d) $V \approx 2 \cdot 2\pi \int_{0}^{0.89} x(\cos x - x^4) dx$
≈ 2.99

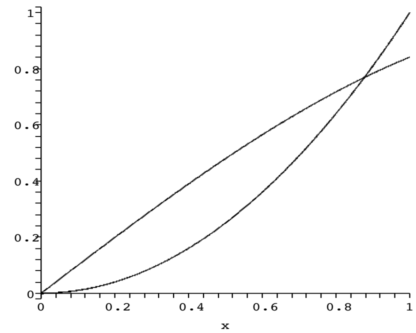


26. (a) $V \approx \pi \int_{0}^{0.85} [(1-x^2)^2 - (1-\sin x)^2] dx$
≈ 0.57

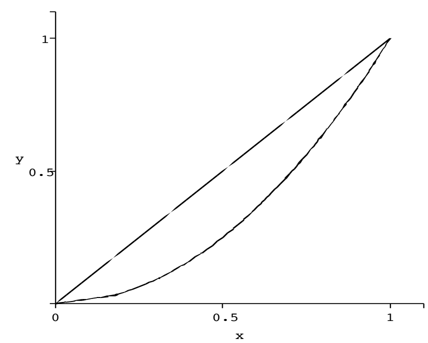
(b) $V \approx 2\pi \int_{0}^{0.85} (1-x)\cdot(\sin x - x^2) dx$
≈ 0.47

(c) $V \approx 2\pi \int_{0}^{0.85} x(\sin x - x^2) dx$
≈ 0.38

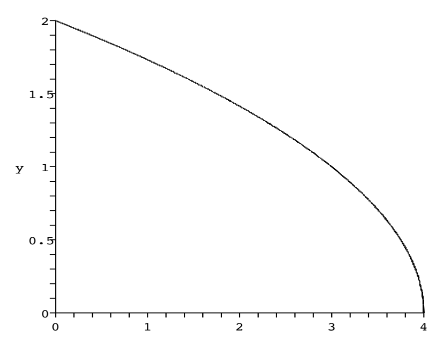
(d) $V \approx \pi \int_{0}^{0.85} [(\sin x)^2 - (x^2)^2] dx$
≈ 0.28



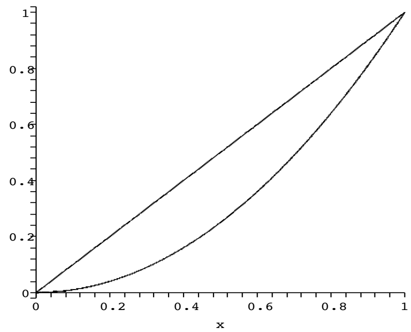
27. Axis of revolution: y-axis
Region bounded by: $x = \sqrt{y}, x = y$



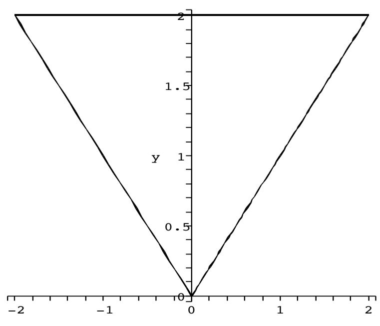
28. Axis of revolution: y-axis
Region bounded by:
$x = 4 - y^2, x = 0, y = 0$



29. Axis of revolution: y-axis
Region bounded by: $y = x, y = x^2$



30. Axis of revolution: $y = 4$
Region bounded by:
$y = x, y = -x, y = 2$



31. If the r-interval $[0, R]$ is partitioned by points r_i, the circular band

$$\{r_i^2 \le x^2 + y^2 \le r_{i+1}^2\}$$

has approximate area $c(r_i)\Delta r_i$ (length times thickness). The limit of the sum of these areas is $A = \lim \sum_{i=1}^{n} c(r_i)\Delta r_i = \int_0^R c(r)dr$ Because we know that $c(r) = 2\pi r$,
we can evaluate the integral, getting
$2\pi \dfrac{r^2}{2}\Big|_0^R = \pi R^2$.

32. If we think of the area of a circle of radius R as being built up as described in Problem 61, then
$A = \displaystyle\int_0^R 2\pi r dr$ Viewed as a function of R, the derivative is
$\dfrac{dA}{dR} = 2\pi R$ so this is, of course, not a coincidence.

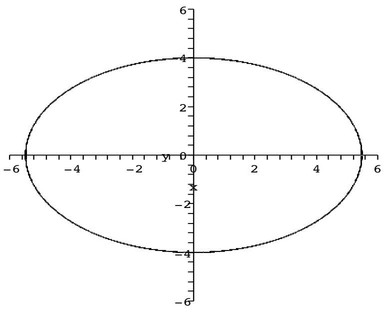
33. The volume that we are looking for is twice the volume of a shell with radius x and height $\sqrt{1-x^2}$.
In other words, The bead is mathematically the solid formed up from revolving the region bounded by $y = \sqrt{1-x^2}, x = 1/2$
and the x-axis around the y-axis.
Therefore
$V = 2 \cdot \displaystyle\int_{1/2}^{1} 2\pi x\sqrt{1-x^2}dx$
Let $u = 1 - x^2, du = -2xdx$,
and $V = 4\pi \displaystyle\int_{1/2}^{1} x\sqrt{1-x^2}dx$
$= -\dfrac{1}{2} 4\pi \displaystyle\int_{3/4}^{0} u^{1/2}du$
$= 2\pi \cdot \dfrac{2}{3} u^{3/2}\Big|_0^{3/4}$
$= \dfrac{\sqrt{3}\pi}{2} cm^3$.

34. The size of the sphere is $4\pi/3$ cm^3, so we look for the value of c such that
$4\pi \displaystyle\int_c^1 x\sqrt{1-x^2}dx = \dfrac{2}{3}\pi$.
$V = 4\pi \displaystyle\int_c^1 x\sqrt{1-x^2}dx$
$= \dfrac{4}{3}\pi(1-c^2)^{3/2} = \dfrac{2}{3}\pi$
Hence we want the size of the hole to be
$c = \sqrt{1 - \sqrt{3}\dfrac{1}{4}} \approx 0.6$ cm.

35. $V = \displaystyle\int_0^1 x(1-x^2)dx$
$= \displaystyle\int_0^1 (x - x^3)dx$
$= \left(\dfrac{x^2}{2} - \dfrac{x^4}{4}\right)\Big|_0^1 = \dfrac{1}{4}$
$V_1 = \displaystyle\int_c^1 x(1-x^2)dx$
$= \left(\dfrac{x^2}{2} - \dfrac{x^4}{4}\right)\Big|_c^1 = \dfrac{1}{4} - \dfrac{c^2}{2} + \dfrac{c^4}{4}$
We want
$V - V_1 = \dfrac{1}{10}V$
Then
$\dfrac{c^2}{2} - \dfrac{c^4}{4} = \dfrac{1}{40}$
$c \approx 0.2265$

36. $V = 4\pi \displaystyle\int_0^4 y\sqrt{30\left(1 - \dfrac{y^2}{16}\right)}dy$
Let $u = 1 - y^2/16, du = -ydy/8$
$V = -32\sqrt{30}\pi \displaystyle\int_1^0 u^{1/2}du$

$$= 32\sqrt{30}\pi \cdot \frac{2}{3} = \frac{64\sqrt{30}\pi}{3}$$



5.4 Arc Length and Surface Area

1. For $n = 2$, the evaluations points are $0, 0.5, 1$
$s \approx s_1 + s_2$
$= \sqrt{(0-0.5)^2 + [f(0) - f(0.5)]^2}$
$+ \sqrt{(1-0.5)^2 + [f(1) - f(0.5)]^2}$
$= \sqrt{0.5^2 + 0.5^4} + \sqrt{0.5^2 + 0.75^2}$
≈ 1.460
For $n = 4$, the evaluations points:
$0, 0, 25, 0.5, 0.75, 1$
$s \approx \sum_{i=1}^{4} s_i \approx 1.474$

2. For $n = 2$, the evaluations points are $0, 0.5, 1$
$s \approx s_1 + s_2 \approx 1.566$
For $n = 4$, the evaluations points:
$0, 0, 25, 0.5, 0.75, 1$
$s \approx \sum_{i=1}^{4} s_i \approx 1.591$

3. For $n = 2$, the evaluations points are
$0, \pi/2, \pi$
$s \approx s_1 + s_2$
$= \sqrt{(\pi/2)^2 + [\cos(\pi/2) - \cos 0]^2}$
$+ \sqrt{(\pi/2)^2 + [\cos \pi - \cos(\pi/2)]^2}$
$= \sqrt{\pi^2 + 4} \approx 3.724$
For $n = 4$, the evaluations points:
$0, \pi/4, \pi/2, 3\pi/4, \pi$
$s \approx \sum_{i=1}^{4} s_i \approx 3.790$

4. For $n = 2$, the evaluation points are $1, 2, 3$
$s \approx s_1 + s_2$
$= \sqrt{1^2 + (\ln 2 - \ln 1)^2}$
$+ \sqrt{1^2 + (\ln 3 - \ln 2)^2}$
≈ 2.296
For $n = 4$, the evaluation points are
$1, 1.5, 2, 2.5, 3$
$s \approx \sum_{i=1}^{4} s_i \approx 4.161$

5. This is a straight line segment from $(0, 1)$ to $(2, 5)$. As such, its length is
$s = \sqrt{(5-1)^2 + (2-0)^2}$
$= \sqrt{20} = 2\sqrt{5}$

6. $s = \int_{-1}^{1} \sqrt{1 + \frac{x^2}{1-x^2}}\, dx$
$= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx$
$= \left(\sin^{-1} x\right)\Big|_{-1}^{1} = \pi$

7. $y'(x) = 6x^{1/2}$, the arc length integrand is $\sqrt{1 + (y')^2} = \sqrt{1 + 36x}$.
Let $u = 1 + 36x$ then
$s = \int_{1}^{2} \sqrt{1 + 36x}\, dx$
$= \int_{37}^{73} \sqrt{u} \left(\frac{du}{36}\right)$
$= \frac{2}{3(36)} u^{3/2}\Big|_{37}^{73}$
$= \frac{1}{54}(73\sqrt{73} - 37\sqrt{37})$
≈ 7.3824

8. $s = \int_{0}^{1} \sqrt{1 + (e^{2x} - e^{-2x})^2}\, dx$
$= \int_{0}^{1} \sqrt{e^{4x} - 1 + e^{-4x}}\, dx$
≈ 3.056

9. $y'(x) = \frac{2x}{4} - \frac{1}{2x} = \frac{1}{2}\left(x - \frac{1}{x}\right)$
$1 + (y')^2 = 1 + \frac{1}{4}\left(x^2 - 2 + \frac{1}{x^2}\right)$
$= \frac{1}{4}\left(x^2 + 2 + \frac{1}{x^2}\right)$
$= \left[\frac{1}{2}\left(x + \frac{1}{x}\right)\right]^2$
$s = \frac{1}{2}\int_{1}^{2} \left(x + \frac{1}{x}\right) dx$

$$= \frac{1}{2}\left(\frac{x^2}{2}+\ln x\right)\Big|_1^2$$
$$= \frac{1}{2}\left(\frac{3}{2}+\ln 2\right)$$
$$\approx 1.0965$$

10. $y'(x) = \frac{1}{2}(x^2+x^{-2})$
$$s = \int_1^3 \sqrt{1+\left(\frac{x^2}{2}+\frac{1}{2x^2}\right)^2}dx$$
$$= \frac{1}{2}\int_1^3 \frac{\sqrt{x^8+6x^4+1}}{x^2}dx$$
$$\approx 5.152$$

11. $x'(y) = \frac{y^3}{2}-\frac{1}{2y^3} = \frac{1}{2}\left(y^3-\frac{1}{y^3}\right)$
$$1+(x')^2 = 1+\frac{1}{4}\left(y^6-2+\frac{1}{y^6}\right)$$
$$= \frac{1}{4}\left(y^6+2+\frac{1}{y^6}\right)$$
$$= \left[\frac{1}{2}(y^3+\frac{1}{y^3})\right]^2$$
$$s = \int_{-2}^{-1}\sqrt{1+(x')^2}\,dy$$
$$= -\frac{-1}{-2}\int_{-2}^{-1}\left(y^3+\frac{1}{y^3}\right)dy$$
$$= \frac{1}{2}\left(-\frac{y^4}{4}\Big|_{-2}^{-1}+\frac{1}{2y^2}\Big|_{-2}^{-1}\right)$$
$$= \frac{1}{2}\left(\frac{15}{4}+\frac{3}{8}\right) = \frac{33}{16}$$

12. Here $x(y) = e^{y/2}+e^{-y/2}$
$x'(y) = \frac{1}{2}\left(e^{y/2}-e^{-y/2}\right)$
Now
$$s = \int_{-1}^{1}\sqrt{1+\left[\frac{1}{2}\left(e^{y/2}-e^{-y/2}\right)\right]^2}dy$$
$$= \frac{1}{2}\int_{-1}^{1}\left(e^{y/2}+e^{-y/2}\right)dy$$
$$= \int_0^1\left(e^{y/2}+e^{-y/2}\right)dy$$
$$= 2\left(e^{y/2}-e^{-y/2}\right)\Big|_0^1 = 2\left(\frac{e-1}{\sqrt{e}}\right)$$

13. $y'(x) = \frac{x^{1/2}}{2}-\frac{x^{-1/2}}{2}$
$$= \frac{1}{2}\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)$$
$$1+(y')^2 = 1+\frac{1}{4}\left(x-2+\frac{1}{x}\right)$$
$$= \frac{1}{4}\left(x+2+\frac{1}{x}\right)$$
$$= \left[\frac{1}{2}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)\right]^2$$
$$s = \int_1^4\sqrt{1+(y')^2}$$
$$= \frac{1}{2}\int_1^4\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$$
$$= \frac{x^{3/2}}{3}\Big|_1^4+\sqrt{x}\Big|_1^4$$
$$= \frac{7}{3}+1 = \frac{10}{3}$$

14. Here $f(x) = 2\ln\left(4-x^2\right)$
$\Rightarrow f'(x) = \dfrac{-4x}{(4-x^2)}$
$$1+\left(f'(x)\right)^2 = 1+\left(\frac{-4x}{(4-x^2)}\right)^2 = \left(\frac{4+x^2}{4-x^2}\right)^2$$
Now , $s = \displaystyle\int_0^1\left(\frac{4+x^2}{4-x^2}\right)dx = 2\ln(3)-1$

15. $s = \displaystyle\int_{-1}^{1}\sqrt{1+(3x^2)^2}dx$
$$= \int_{-1}^{1}\sqrt{1+9x^4}dx \approx 3.0957$$

16. $s = \displaystyle\int_{-2}^{2}\sqrt{1+9x^4}dx \approx 17.2607$

17. $s = \displaystyle\int_0^2\sqrt{1+(2-2x)^2}dx \approx 2.9578$

18. $s = \displaystyle\int_0^{\pi/4}\sqrt{1+\sec^4 x}dx \approx 1.2780$

19. $s = \displaystyle\int_0^{\pi}\sqrt{1+(-\sin x)^2}dx$
$$= \int_0^{\pi}\sqrt{1+\sin^2 x}dx \approx 3.8201$$

20. $s = \displaystyle\int_1^3\sqrt{1+\frac{1}{x^2}}dx \approx 2.3020$

21. $s = \displaystyle\int_0^{\pi}\sqrt{1+(x\sin x)^2}dx = 4.6984$

22. $s = \displaystyle\int_0^{\pi}\sqrt{1+e^{-x}\sin^2 x}dx \approx 13.1152$

23. Here $f(x) = 10\left(e^{x/20} + e^{-x/20}\right)$

$\Rightarrow f'(x) = \dfrac{10}{20}\left(e^{x/20} - e^{-x/20}\right)$

$1 + \left(f'(x)\right)^2 = 1 + \left(\dfrac{1}{2}\left(e^{x/20} - e^{-x/20}\right)\right)^2$

$= \left(\dfrac{1}{2}\left(e^{x/20} + e^{-x/20}\right)\right)^2$

Now,

$s = \displaystyle\int_{-20}^{20} \frac{1}{2}\left(e^{x/20} + e^{-x/20}\right) dx$

$= \displaystyle\int_{0}^{20} \left(e^{x/20} + e^{-x/20}\right) dx$

$= 20\left(e^{x/20} - e^{-x/20}\right)\Big|_0^{20}$

$= 20\left(e - e^{-1}\right) \approx 47.0080$

24. $s = \displaystyle\int_{-30}^{30} \sqrt{1 + \left[\frac{1}{2}\left(e^{x/30} - e^{-x/30}\right)\right]^2}\, dx$

$= \displaystyle\int_{-30}^{30} \frac{1}{2}\left(e^{x/30} + e^{-x/30}\right) dx$

$= \left(15e^{x/30} - 15e^{-x/30}\right)\Big|_{-30}^{30}$

$= 30e - 30e^{-1} \approx 70.51207161 ft.$

25. In Example 4.4, $y(x) = 5(e^{x/10} + e^{-x/10})$

$y(0) = 5(e^0 + e^0) = 10$

$y(-10) = y(10)$

$= 5(e^1 + e^{-1}) = 15.43$

sag $= 15.43 - 10 = 5.43$ ft

A lower estimate for the arc length given the sag would be

$2\sqrt{(10)^2 + (sag)^2}$

$= 2\sqrt{100 + 29.4849} \approx 22.76$

This looks good against the calculated arc length of 23.504.

26. If $x^{2/3} + y^{2/3} = 1$, then in the first quadrant, $y = (1 - x^{2/3})^{3/2}$ and taking only the first-quadrant case (which would produce one fourth of the total length s), we have $y = \dfrac{3}{2}(1 - x^{2/3})^{1/2}\left(-\dfrac{2}{3}x^{-1/3}\right)$

$= -x^{-1/3}(1 - x^{2/3})^{1/2}$

$(y')^2 = x^{-2/3}(1 - x^{2/3}) = x^{-2/3} - 1$

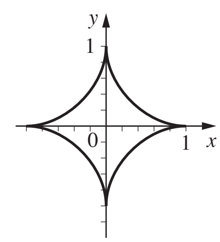
$s = 4\displaystyle\int_0^1 \sqrt{1 + y'^2}\, dx$

$= 4\displaystyle\int_0^1 \sqrt{x^{-2/3}}\, dx$

$= 4\displaystyle\int_0^1 x^{-1/3}\, dx$

$= 4\left(\dfrac{3}{2}\right) x^{2/3}\Big|_0^1 = 6$

There are some technicalities in fully justifying the preceding computation, since the integrand $(x^{-1/3})$ is unbounded at $x = 0$, but the conclusion is sound.



27. $y = 0$ when $x = 0$ and when $x = 60$, so the punt traveled 60 yards horizontally.

$y'(x) = 4 - \dfrac{2}{15}x = \dfrac{2}{15}(30 - x)$

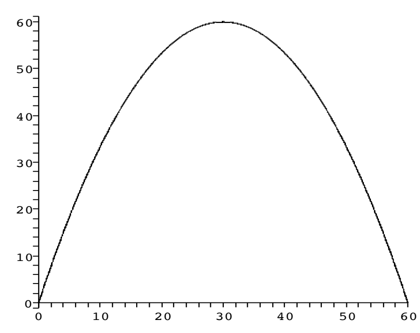
This is zero only when $x = 30$, at which point the punt was $(30)^2/15 = 60$ yards high.

$s = \displaystyle\int_0^{60} \sqrt{1 + \left(4 - \frac{2}{15}x\right)^2}\, dx$

≈ 139.4 yards

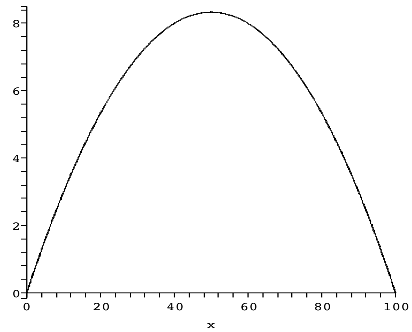
$v = \dfrac{s}{4 \text{ sec}} = \dfrac{139.4 \text{ yards}}{4\,\text{sec}} \cdot \dfrac{3 \text{ feet}}{1 \text{ yard}}$

$= 104.55$ ft/s



28. Since $y(100) = 0$, the ball traveled 100 yards. The maximum height of the ball is $y(50) = \dfrac{25}{3}$ yards. The arc length is $s = \displaystyle\int_0^{100} \sqrt{1 + \left[\frac{1}{300}(100 - 2x)\right]^2}\, dx$

≈ 101.82215 yards



29. $S = 2\pi \int_0^1 y\, ds$
$= 2\pi \int_0^1 x^2 \sqrt{1+(2x)^2} dx$
≈ 3.8097

30. $S = \int_0^\pi 2\pi \sin x \sqrt{1+\cos^2 x} dx$
≈ 14.42360

31. $S = 2\pi \int_0^2 y\, ds$
$= 2\pi \int_0^2 (2x-x^2)\sqrt{1+(2-2x)^2} dx$
≈ 10.9654

32. $S = \int_{-2}^0 2\pi(x^3-4x)\sqrt{1+(3x^2-4)^2} dx$
≈ 67.06557

33. $S = 2\pi \int_0^1 y\, ds$
$= 2\pi \int_0^1 e^x \sqrt{1+e^{2x}} dx \approx 22.9430$

34. $S = \int_1^2 2\pi \ln x \sqrt{1+\frac{1}{x^2}} dx$
≈ 2.86563

35. $S = 2\pi \int_0^{\pi/2} y\, ds$
$= 2\pi \int_0^{\pi/2} \cos x \sqrt{1+\sin^2 x} dx$
≈ 7.2117

36. $S = \int_1^2 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}} dx \approx 8.28315$

37. $s_1 = \int_0^1 \sqrt{1+(6x^5)^2} dx$
$= \int_0^1 \sqrt{1+36x^{10}} dx \approx 1.672$
$s_2 = \int_0^1 \sqrt{1+(8x^7)^2} dx$
$= \int_0^1 \sqrt{1+64x^{14}} dx \approx 1.720$
$s_3 = \int_0^1 \sqrt{1+(10x^9)^2} dx$
$= \int_0^1 \sqrt{1+100x^{18}} dx \approx 1.75$
As $n \to \infty$, the length approaches 2, since one can see that the graph of $y = x^n$ on $[0,1]$ approaches a path consisting of the horizontal line segment from $(0,0)$ to $(1,0)$ followed by the vertical line segment from $(1,0)$ to $(1,1)$.

38. (a) For $0 \le x < 1$, we have $\lim_{n\to\infty} x^n = 0$ Therefore, the length of the limiting curve is 1 (the limiting curve is a horizontal line). Connecting the limiting curve to the endpoint at $(1,1)$ adds an additional length of 1 for a total length of 2.

(b) $y_1 = x^4, y_1' = 4x^3$
$y_2 = x^2, y_2' = 2x$
Since both are increasing for positive x, y_1 is "steeper" (y_2 is "flatter") if and only if $y_1' > y_2'$, i.e.,
$4x^3 > 2x,\ x^2 > \frac{1}{2},\ x > \sqrt{\frac{1}{2}}$

39. (a) $L_1 = \int_{-\pi/6}^{\pi/6} \sqrt{1+\cos^2 x} dx \approx 1.44829$

$L_2 = \sqrt{\left(\sin\frac{\pi}{6} - \sin\left(-\frac{\pi}{6}\right)\right)^2 + \left(\frac{2\pi}{6}\right)^2}$
≈ 1.44797 Hence

$\frac{L_2}{L_1} = \frac{1.44797}{1.44829} \approx .9998$

(b) $L_1 = \int_{-\pi/2}^{\pi/2} \sqrt{1+\cos^2 x} dx \approx 3.8202$
$L_2 = \sqrt{\left(2\sin\frac{\pi}{2}\right)^2 + (\pi)^2}$
$= \sqrt{\pi^2+4} = 3.7242$
Hence
$\frac{L_2}{L_1} \approx 0.9749$

40. (a) $L_1 = \int_3^5 \sqrt{1+(e^x)^2} dx \approx 128.3491$

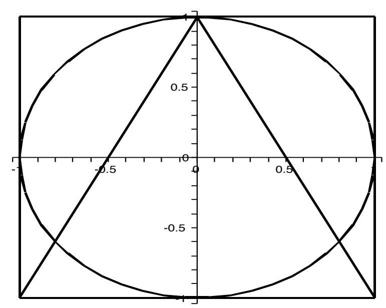
$L_2 = \sqrt{2^2+(e^5-e^3)^2} \approx 128.3432$
Hence
$\frac{L_2}{L_1} \approx 0.9999$

(b) $L_1 = \int_{-5}^{-3} \sqrt{1+(e^x)^2}dx \approx 2.0006$

$L_2 = \sqrt{2^2+(e^{-5}-e^{-3})^2} \approx 2.0005$

Hence

$\dfrac{L_2}{L_1} \approx 0.9999$



41. (a) Considering only the vertical segment $x = 1$, $(-1 < y < 1)$, the area after rotation, as an integral in y, would be

$$2\pi \int_{y=-1}^{y=1} x ds(y) = 2\pi \int_{-1}^{1} (1)\sqrt{1+0^2}dy$$

$= 2\pi y|_{-1}^{1} = 4\pi$

(height times circumference)

The full solid of revolution is a cylinder with radius 1, and its top and bottom each have area $\pi(1)^2 = \pi$. Hence the total surface area is $4\pi + \pi + \pi = 6\pi$.

(b) $S = \displaystyle\int_{-1}^{1} 2\pi\sqrt{1-y^2}\sqrt{1+\left(\frac{y}{\sqrt{1-y^2}}\right)^2} dy$

$= \displaystyle\int_{-1}^{1} 2\pi\sqrt{1-y^2}\sqrt{\frac{1}{\sqrt{1-y^2}}}\,dy$

$= \displaystyle\int_{-1}^{1} 2\pi\,dy = 4\pi$

(c) The equation for the right segment of the triangle is $x = (1-y)/2$. Hence the resulting area is $2\pi \displaystyle\int_{y=-1}^{y=1} x ds(y)$

$= 2\pi \displaystyle\int_{-1}^{1} \left(\frac{1-y}{2}\right)\sqrt{1+\left(-\frac{1}{2}\right)^2} dy$

$= 2\pi \displaystyle\int_{-1}^{1} \left(\frac{1-y}{2}\right)\sqrt{\frac{5}{4}}dy$

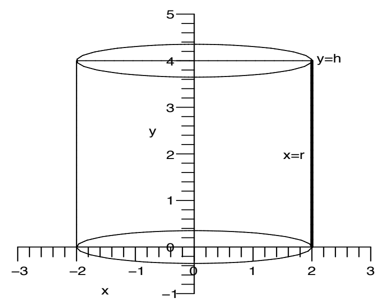
$= \dfrac{\pi\sqrt{5}}{2}\left(y - \dfrac{y^2}{2}\right)\Big|_{-1}^{1} = \pi\sqrt{5}$

The full revolved figure is a cone with added base of radius 1 (and area π). Hence the total surface area

$\pi\sqrt{5} + \pi(\sqrt{5}+1)\pi$.

(d) $6\pi : 4\pi : (\sqrt{5}+1)\pi = 3 : 2 : \tau$

42. (a) Surface area of a right circular cylinder of radius r and height h.



Consider a line $x = r$ and $0 \le y \le h$ rotating about the $y - axis$ to form a Right Circular Cylinder.

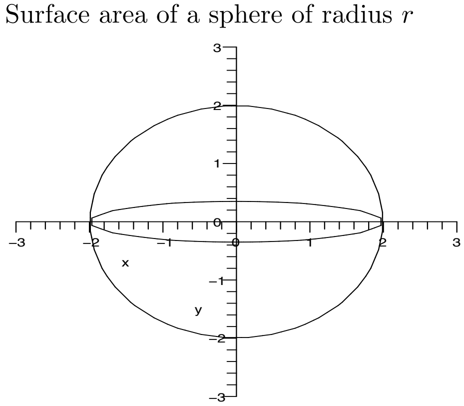
Here $f(y) = r$

Therefore, the surface area

$$S = \int_0^h 2\pi f(y)\sqrt{1+(f'(y))^2}dy$$

$$= \int_0^h 2\pi r\sqrt{1+(0)^2}dy = 2\pi rh$$

(b) Surface area of a sphere of radius r



Consider a semicircle of radius r with centre as the origin, its equation is $y = \sqrt{r^2 - x^2}$

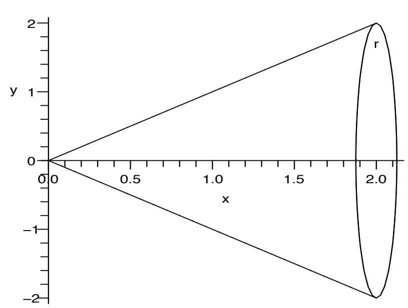
for $-r \leq x \leq r$ Rotating it about the $x-axis$ we get a sphere Here

$f(x)=\sqrt{r^2-x^2}$

Therefore, the surface area

$$S=2\pi\int_{-r}^{r} f(x)\sqrt{1+(f'(x))^2}dx$$
$$=2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}dx$$
$$=2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\sqrt{1+\frac{x^2}{r^2-x^2}}dx$$
$$=2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\sqrt{\frac{r^2-x^2+x^2}{r^2-x^2}}dx$$
$$=2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\sqrt{\frac{r^2}{r^2-x^2}}dx$$
$$=2\pi\int_{-r}^{r} r\,dx$$
$$=4\pi r^2$$

(c) Surface area of cone of radius r and height h



Consider a line $y=\left(\frac{r}{h}\right)x$ Rotating it about the $x-axis$, we get a cone of radius r and height h Here

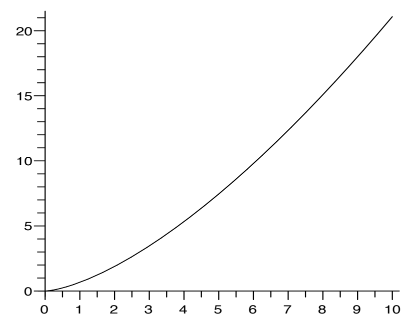
$f(x)=\left(\frac{r}{h}\right)x$

Therefore, the surface area

$$S=2\pi\int_{0}^{h} f(x)\sqrt{1+(f'(x))^2}dx$$
$$=2\pi\int_{0}^{h}\frac{rx}{h}\sqrt{1+\left(\frac{r}{h}\right)^2}dx$$
$$=2\pi\int_{0}^{h}\frac{rx}{h}\sqrt{\frac{r^2+h^2}{h^2}}dx$$
$$=2\pi\int_{0}^{h}\frac{rx}{h^2}\sqrt{r^2+h^2}dx$$
$$=\frac{2\pi r\sqrt{r^2+h^2}}{h^2}\left(\frac{x^2}{2}\right)\Big|_0^h$$
$$=\pi r\sqrt{r^2+h^2}=\pi rl$$

where $l=\sqrt{r^2+h^2}$ is the slanted height of the cone.

43.



For the path along the positive $x-axis$, the equation of the path is $f(x)=0$ Therefore $f'(x)=0$ The distance covered along the $x-axis$ is

$$L_1=\int_0^s\sqrt{1+f'(x)}dx=\int_0^s dx\Rightarrow L_1=s$$

Now, for the path along the curve

$y=\frac{2}{3}(x)^{3/2}$

The equation of the path is

$f(x)=\frac{2}{3}(x)^{3/2}$

Therefore

$f'(x)=\frac{2}{3}\cdot\frac{3}{2}\cdot x^{1/2}\Rightarrow f'(x)=x^{1/2}$

The distance covered along these curve is

$$L_2=\int_0^s\sqrt{1+f'(x)}dx=\int_0^s\sqrt{1+x}\,dx$$
$$L_2=\frac{2}{3}(s+1)^{3/2}-\frac{2}{3}$$

(a) Consider $L_2=2L_1$

$$\frac{L_2}{L_1}=\frac{2(s+1)^{\frac{3}{2}}-2}{3s}=2$$
$$\Rightarrow(s+1)^{\frac{3}{2}}=3s+1 \text{ or}$$
$$(s+1)^3=(3s+1)^2$$
$$\Rightarrow s^3-6s^2-3s=0$$

Thus $s=0$ or $s=6.464102$ or $s=-0.464102$

But $s>0$, therefore $s=6.464102$

(b) Consider the motion of the person along the $x - axis$
Let $g(t)$ be the distance walked along the $x - axis$
Therefore $g(t) = t, 0 \le t \le x, \Rightarrow g'(t) = 1$
Now, consider the motion of the person along the curve $y = \frac{2}{3}(x)^{3/2}$
$f(t) = \frac{2}{3}(t)^{3/2}$ is the distance walked along the curve $y = \frac{2}{3}(x)^{3/2}, 0 \le t \le x$
Therefore
$f(t) = \frac{2}{3}(t)^{3/2}, 0 \le t \le x \Rightarrow f'(t) = \sqrt{t}$
The ratio of the speeds $= \frac{f'(t)}{g'(t)} = \frac{\sqrt{t}}{1} = 2$
$\Rightarrow t = 4$

44. (a) $\frac{d}{dx}\sqrt{2}\int_0^x \sqrt{1 - \frac{\sin^2 u}{3}}du$
$= \frac{1}{2}\sqrt{2} \cdot \sqrt{4 - 2\sin^2 x}$
$= \sqrt{1 + \cos^2 x}$

(b) $\frac{d}{dx}\left(\frac{1}{4}x\sqrt{1+16x^6} + \int \frac{3/4}{\sqrt{1+16x^6}}\,dx\right)$
$= \left(\frac{1}{4}\sqrt{1+16x^6} + \frac{12x^6}{\sqrt{1+16x^6}}\right) + \frac{3/4}{\sqrt{1+16x^6}}$
$= \frac{1/4(1+16x^6)}{\sqrt{1+16x^6}} + \frac{12x^6}{\sqrt{1+16x^6}} + \frac{3/4}{\sqrt{1+16x^6}}$
$= \frac{1+16x^6}{\sqrt{1+16x^6}} = \sqrt{1+16x^6}$

5.5 Projectile Motion

1. $y(0) = 80, y'(0) = 0$

2. $y(0) = 100, y'(0) = 0$

3. $y(0) = 60, y'(0) = 10$

4. $y(0) = 20, y'(0) = -4$

5. The initial conditions are
$y(0) = 30$ and $y'(0) = 0$
We want to find $y'(t)$ when $y(t) = 0$.

We start with the equation $y''(t) = -32$.
Integrating gives $y'(t) = -32t + c_1$.
From the initial velocity, we have
$0 = y'(0) = -32(0) + c_1$, and so $y'(t) = -32t$
Integrating again gives $y(t) = -16t^2 + c_2$.
From the initial position, we have
$30 = y(0) = -16(0) + c_2$ and so
$y(t) = -16t^2 + 30$.

Solving $y(t) = 0$ gives $t = \pm\sqrt{\frac{15}{8}}$ The positive solution is the solution we are interested in. This is the time when the diver hits the water. The diver's velocity is therefore
$y'\left(\sqrt{\frac{15}{8}}\right) = -32\sqrt{\frac{15}{8}} \approx -43.8$ ft/sec

6. The initial conditions are
$y(0) = 120$ and $y'(0) = 0$
We want to find $y'(t)$ when $y(t) = 0$. We start with the equation $y''(t) = -32$.
Integrating gives $y'(t) = -32t + c_1$.
From the initial velocity, we have
$0 = y'(0) = -32(0) + c_1$, and so $y'(t) = -32t$.

Integrating again gives $y(t) = -16t^2+c_2$. From the initial position, we have
$120 = y(0) = -16(0) + c_2$ and so
$y(t) = -16t^2 + 120$.

Solving $y(t) = 0$ gives $t = \pm\sqrt{\frac{15}{2}}$. The positive solution is the solution we are interested in. This is the time when the diver hits the water. The diver's velocity is therefore
$y'\left(\sqrt{\frac{15}{2}}\right) = -32\sqrt{\frac{15}{2}}$ ft/sec

7. If an object is dropped (time zero, zero initial velocity) from an initial height of y_0, then the impact moment is $t_0 = \sqrt{y_0}/4$ and the impact velocity (ignoring possible negative sign) is $v_{\text{impact}} = 32t_0 = 8\sqrt{y_0}$
Therefore if the object is dropped from 30 ft, the impact velocity is
$8\sqrt{30} \approx 43.8178$ feet per second.
If dropped from 120 ft, impact velocity is
$8\sqrt{120} \approx 87.6356$ feet per second.
From 3000 ft, impact velocity is
$8\sqrt{3000} \approx 438.178$ feet per second.
From a height of $h\, y_0$, the impact velocity is
$8\sqrt{hy_0} = 8\sqrt{h}\sqrt{y_0} = \sqrt{h}\left(8\sqrt{y_0}\right)$,
which is to say that impact velocity increases by a factor of $\sqrt{h}$ when initial height increases by a factor of h.

8. Ignoring air friction we have initial conditions
$y(0) = 555.427$ and $y'(0) = 0$.

Integrating $y''(t) = -32$ gives
$y'(t) = -32t + c_1$. The initial condition gives

$0 = y'(0) = -32(0) + c_1$ and therefore
$y'(t) = -32t.$

Integrating again gives $y(t) = -16t + c_2$.

The initial condition gives
$555.427 = y(0) = -16(0) + c_2$ and therefore
$y(t) = -16t^2 + 555.427.$

We will assume that the baseball player catches the ball when it is 6 feet above the ground, so we solve
$6 = y(t) = -16t^2 + 555.427$. Solving gives $t \approx \pm 5.86$. We use the positive solution.
The velocity at this time is
$y'(5.86) = -16(5.86) = -93.75$ ft/sec
(If you assume the ball is caught at ground level, the ball will be going 94.27 ft/sec.)

9. As $y''(t) = -9.8, y'(t) = -9.8t + y'(0)$

Therefore, $y(t) = -4.9t^2 + y'(0)\,t + y(0)$

where $y(0)$ represents the height of the cliff and $y(4) = 0$.
Now, $y(4) = -4.9\,(16) + 4\,(0) + y\,(0)$

Thus, $y(0) = 78.4$ is the height of the cliff in meters.

10. Let $y(t)$ be the height of the boulder.
Therefore $y''(t) = -9.8; y(3) = 0$ and
$y'(0) = 0$
Thus, $y'(t) = -9.8t + y'(0)$ and
$y(t) = -4.9t^2 + y'(0)\,t + y(0)$
Thus,
$y(3) = -4.9\,(9) + y\,(0) \Rightarrow y\,(0) = 43.1$meters

11. Let $y(t)$ be the height at any time t.
Here $v'(t) = -9.8$
Therefore $v(t) = -9.8\,t + v(0) = -9.8t + 19.6$
or $y'(t) = -9.8\,t + 19.6$
$\Rightarrow y(t) = -4.9t^2 + 19.6\,t + y(0)\,.$

But $y(0) = 0$ therefore, $y(t) = -4.9t^2 + 19.6\,t$ which is the height at ay time t. Also the velocity at any instant t is
$v(t) = -9.8\,t + 19.6 = -9.8\,(t-2)$

Now for the maximum height,
$v(t) = 0 \Rightarrow t = 2.$
Therefore, maximum height is
$y(2) = -4.9(2)^2 + 19.6\,(2) + y\,(0) = 19.6$

He remains in the air until $y(t) = 0$.
That is, till $-4.9t^2 + 19.6t = 0 \Rightarrow t = 0$ or $t = 4$
Therefore, the amount of time he spent in the air is 4sec.
The velocity with which he smacks back is
$v(4) = -9.8\,(4-2) = -19.6\,\mathrm{m/s}$

12. Let $y(t)$ be the height at any time t.
Here $v'(t) = -9.8,$
Therefore $v(t) = -9.8t + v(0)$
$\Rightarrow y'(t) = -9.8\,t + v(0)$
$\Rightarrow y(t) = -4.9t^2 + v(0)\;t + y(0)\,.$

But $y(0) = 0$.
Therefore, $y(t) = -4.9t^2 + v(0)\;t$ which is the height at any time t.
Now the maximum height is reached when
$y'(t) = 0$ that is when $t = \dfrac{v(0)}{9.8}$.
Therefore for the maximum height

$$y\left(\frac{v(0)}{9.8}\right) = -4.9\left(\frac{v(0)}{9.8}\right)^2 + v(0)\,\left(\frac{v(0)}{9.8}\right)$$

$$\Rightarrow 78.4 = -4.9\left(\frac{v(0)}{9.8}\right)^2 + v(0)\,\left(\frac{v(0)}{9.8}\right)$$

$$\Rightarrow \frac{(v(0))^2}{9.8}\left[-\frac{4.9}{9.8} + 1\right] = 78.4$$

$\Rightarrow v(0) = 39.2\,\mathrm{m/s}$

13. Reviewing the solution to Exercise 11, the difference is that $v(0)$ is unknown. However, we still see that
$y = -16t^2 + tv(0) = -t[16t - v(0)]$ (factoring, rather than completing the square). The second time that $y = 0$ can be seen to occur at time $t_2 = v(0)/16$, at which time
$v(t_2) = -32t_2 + v(0) = v(0)(-2+1) = -v(0)$

Now we see
$v(t) = -32t + v(0) = -32t + 16t_2$
$= -16(2t - t_2)$

The peak was therefore at time $t_2/2$, at which time the height was $-(t_2/2)[16t2/2 - v(0)]$
$= -(t2/2)[(v(0)/2) - v(0)]$
$= -(v(0)/32)[-v(0)/2] = v(0)^2/64.$

In summary, $y_{\max} = [v(0)/8]^2$ in this problem (and more generally, $y_{\max} = [v(0)/8]^2 + y(0)$).
If $y_{\max} = 20$ inches $= 5/3$ feet, then
$v(0)/8 = \sqrt{5/3}$, and
$v(0) = 8\sqrt{5/3} \approx 10.33$ feet per second.

This is considerably less than Michael Jordan's initial velocity of about 17 feet per second, but the difference in velocity is not as dramatic as in height (20 inches to 54 inches).

14. For a given initial velocity of v_0, the velocity and position are given by
$y' = -32t + v_0$
$y = -16t^2 + v_0 t$

The maximum occurs when $y' = 0$ or when

$t_0 = \dfrac{v_0}{32}$
and the maximum height is
$y(t_0) = -16\left(\dfrac{v_0}{32}\right)^2 + v_0\left(\dfrac{v_0}{32}\right) = \left(\dfrac{v_0}{8}\right)^2$
Therefore if the new initial velocity was $1.1v_0$ (an increase of 10%), the new maximum height would be
$\left(\dfrac{1.1v_0}{8}\right)^2 = 1.21\left(\dfrac{v_0}{8}\right)^2$
In other words, it would be an increase in height by 21%.

15. (a) If the initial conditions are
$y(0) = H$ and $y'(0) = 0$
Integrating $y''(t) = -32$ gives
$y'(t) = -32t + c_1$.
The initial condition gives
$y'(t) = -32t + v_0 = -32t$.
Integrating gives
$y(t) = -16t^2 + c_2$.
The initial condition gives
$y(t) = -16t^2 + H$.
The impact occurs when $y(t_0) = 0$ or when $t_0 = \sqrt{y_0}/4 = \sqrt{H}/4$. Therefore the impact velocity is
$y'(t_0) = -32t_0 = -8\sqrt{H}$

(b) If the initial conditions are
$y(0) = 0$ and $y'(0) = v_0$
Integrating $y''(t) = -32$ gives
$y'(t) = -32t + c_1$.
The initial condition gives
$y'(t) = -32t + v_0$.
Integrating gives
$y(t) = -16t^2 + v_0 t + c_2$.
The initial condition gives
$y(t) = -16t^2 + v_0 t$.
The maximum occurs when $y'(t) = 0$ or when $t = v_0/32$.
Therefore the maximum height is
$y\left(\dfrac{v_0}{32}\right) = -\dfrac{16v_0^2}{32^2} + \dfrac{v_0^2}{32} = \dfrac{v_0}{64}$.

16. (a) The time t_0 when the lead ball hits the ground satisfies
$179 = 12800 \ln\left(\cosh\left(\dfrac{t_0}{20}\right)\right)$
$\cosh\left(\dfrac{t_0}{20}\right) = e^{179/12800}$
$t_0 \approx 3.3526$
At time t_0, the height of the wood ball is
$179 - \dfrac{7225}{8}\ln\left(\cosh\left(\dfrac{16}{85}t_0\right)\right)$
$\approx 179 - 169.0337 = 9.9663$ ft

(b) The time t_1 that the wood ball need to hit the ground satisfies
$179 = \dfrac{7225}{8}\ln\left(\cosh\left(\dfrac{16}{85}t_1\right)\right)$
$\cosh\left(\dfrac{16}{85}t_1\right) = e^{1432/7225}$
$t_1 \approx 3.4562$
The wood ball need to be released about $t_1 = t_0 = 0.1036$ seconds earlier.

17. The starting point is
$y'' = -9.8$, $y'(0) = 98\sin(\pi/3) = 49\sqrt{3}$.
We get $y(t) = -4.9t^2 + ty'(0)$
$= -4.9t(t - [v(0)/4.9])$
$= -4.9t(t - 10\sqrt{3})$
The flight time is $10\sqrt{3}$. As to the horizontal range, we have $x'(t)$ constant and forever equal to $98\cos(\pi/3) = 49$. Therefore $x(t) = 49t$ and in this case, the horizontal range is $49(10\sqrt{3})$ (meters).

18. Here $y'(0) = 40\sin\left(\dfrac{\pi}{6}\right) = 20$
Therefore $y(t) = -4.9t^2 + 20t$
$= t(-4.9t + 20)$
$\Rightarrow$ the time of flight $= t = \dfrac{20}{4.9} = 4.082$
Now, for the horizontal range $x(t)$
$x'(t) = 40\cos\left(\dfrac{\pi}{6}\right) = 20\sqrt{3}$
Therefore
$x(t) = 20\sqrt{3}t$ and
$x(4.082) = 20(1.7321)(4.082) = 141.3919$
Repeating the same for the angle 60^0
$y'(0) = 40\sin\left(\dfrac{\pi}{3}\right) = 34.6410$
Therefore
$y(t) = -4.9t^2 + (34.6410)t$
$\Rightarrow y(t) = t(-4.9t + 34.6410)$
$\Rightarrow$ the time of flight $= t = \dfrac{34.6410}{4.9} = 7.0696$
Now, for the horizontal range $x(t)$
$x'(t) = 40\cos\left(\dfrac{\pi}{3}\right) = 20$
Therefore $x(t) = 20t$ and
$x(7.0696) = 20(7.0696) = 141.3919$

19. This problem modifies Example 5.5 by using a service angle of 6° (where the Example 5.5 used 7°) and no other changes. Here the serve hits the net.

Next we want to find the range for which the serve will be in.

If θ is the angle, then the initial conditions are
$x'(0) = 176\cos\theta$, $x(0) = 0$
$y'(0) = 176\sin\theta$, $y(0) = 10$

Integrating $x''(t) = 0$ and $y''(t) = -32$, then using the initial conditions gives
$x'(t) = 176\cos\theta$
$x(t) = 176(\cos\theta)t$
$y'(t) = -32t + 176\sin\theta$
$y(t) = -16t^2 + 176(\sin\theta)t + 10$

To make sure the serve is in, we see what happens at the net and then when the ball hits the ground. First, the ball passes the net when $x = 39$ or when $39 = 176(\cos\theta)t$. Solving gives $t = \dfrac{39}{176\cos\theta}$ Plugging this in for the function $y(t)$ gives

$$y\left(\frac{39}{176\cos\theta}\right)$$
$$= -16\left(\frac{39}{176\cos\theta}\right)^2 + 176(\sin\theta)\left(\frac{39}{176\cos\theta}\right) + 10$$
$$= -\frac{1521}{1936}\sec^2\theta + 39\tan\theta + 10$$

We want to ensure that this value is greater than 3 so we determine the values of θ that give $y > 3$ (using a graphing calculator or CAS). This restriction means that we must have
$-0.15752 < \theta < 1.5507$

Next, we want to determine when the ball hits the ground. This is when
$0 = y(t) = -16t^2 + 176(\sin\theta)t + 10$
We solve this equation using the quadratic formula to get
$$t = \frac{-176\sin\theta \pm \sqrt{176^2\sin^2\theta + 640}}{-32}$$
We are interested in the positive solution, so
$$t = \frac{176\sin\theta + \sqrt{176^2\sin^2\theta + 640}}{32}$$
Substituting this in to
$x(t) = 176(\cos\theta)t$ gives
$$x = 44\cos\theta\left(22\sin\theta + \sqrt{484\sin^2\theta + 10}\right)$$

We want to determine the values of θ that ensure that $x < 60$. Using a graphing calculator or a CAS gives $\theta < -0.13429$

Putting together our two conditions on θ now gives the possible range of angles for which the serve will be in:
$-0.15752 < \theta < -0.13429$

20. In these tennis problems, the issue is purely geometric. Time is irrelevant. One can obtain valuable information by eliminating time and writing y as a function of x. For example, with service angle of θ (in degrees below the horizontal), initial speed v_0, and initial height h, one has
$y(t) = -16t^2 - tv_0\sin\theta + h$,
$x(t) = tv_0\cos\theta$, and hence
$$y = f(x) = \frac{-16x^2}{v_0^2\cos^2\theta} - \frac{x\sin\theta}{\cos\theta} + h$$

Now one could put $x = 60$ (the serve would be in if $f(60) < 0$), or put $x = 39$ (the serve would clear the net if $f(39) > 3$. If one were to set $f(60) = 0$ and solve for v_0, one would obtain a critical speed (call it v_1) for the given (h, θ), *above which the serve would be out.* Solving $f(39) = 3$ one would obtain a second critical speed (call it v_2), *below which the serve would hit the net.* Below we tabulate v_1 and v_2 for $h = 10$ and selected values of θ.

In the 7° line, we see that it would be necessary to reduce the service speed to 149ft./sec. to get it in, and the net would not be a problem. The 7.6° line has these interesting features: the service at 176 ft./sec. is out, whereas the service at 170 ft./sec. is in.

h	θ	v_1	v_2
feet	degrees	ft/sec	ft/sec
10	7.0	149.0	105.7
10	7.6	171.5	117.4
10	8.0	193.6	127.8

21. Let $(x(t), y(t))$ be the trajectory. In this case
$y(0) = 6, x(0) = 0$
$y'(0) = 0, x'(0) = 130$
$x''(t) \equiv 0, x'(t) \equiv 130$
$x(t) = 130t$

This is 60 at time $t = 6/13$. Meanwhile,
$y''(t) = -32, y'(t) = -32t$
$y(t) = -16t^2 + 6$
$$y\left(\frac{6}{13}\right) = -16\left(\frac{6}{13}\right)^2 + 6 = \frac{438}{169}$$
$$y\left(\frac{6}{13}\right) \approx 2.59 \text{ ft}$$

22. If the initial speed is now 80 ft/s, the equations become
$x(t) = 80t$
$y(t) = -16t^2 + 6$

The ball crosses home plate when $x = 60$, or when $t = 3/4$. At the home plate, we then have,
$y(3/4) = -16(3/4)^2 + 6 = -3$

In other words, the ball is "under" the ground and the ball hits the ground before reaching home plate.

23. Let $(x(t), y(t))$ be the trajectory. In this case 5° is converted to $\pi/36$ radians.

$y(0) = 5, x(0) = 0$
$y'(0) = 120\sin\frac{\pi}{36} \approx 10.46$
$x'(0) = 120\cos\frac{\pi}{36} \approx 119.54$
$x''(0) \equiv 0$
$x'(t) \equiv 119.54$
$x(t) = 119.54t$
This is 120 when
$t = 120/119.54 = 1.00385\ldots$

Meanwhile,
$y''(t) = -32$
$y'(t) = -32t + 10.46$
$y(t) = -16t^2 + 10.46t + 5$
$y(1.00385) = -16(1.00385)^2$
$+ 10.46(1.00385) + 5$
$y(1.00385) \approx -.62$ ft

24. We are assuming that the height at 120 feet is the same as the release height 5. Let θ be the angle of release (above the horizontal).
We have
$y(t) = -16t^2 + 120t\sin\theta + 5$
$x(t) = 120t\cos\theta$

Thus $x(t)$ will be 120 when $t = 1/cos\theta$, at which time $y(t)$ will be 5 only if
$\dfrac{-16}{\cos^2\theta} + 120\dfrac{\sin\theta}{\cos\theta} = 0$
Hence if $120\sin\theta\cos\theta = 16$
$60\sin 2\theta = 16$
$2\theta = \sin^{-1}(16/60) = .2699\ldots,$
$\theta = .135$ (radians) or about 7.7°

To find the aim, we need the length of the vertical leg of a right triangle with opposite angle 7.7°, and adjacent leg 120 ft. Thus the player should aim
$120\tan(7.7^\circ) \approx 120\tan(.135) \approx 16.2$ ft

above the first baseman's head.

25. (a) Assuming that the ramp height h is the same as the height of the cars, this problem seems to be asking for the initial speed v_0 required to achieve a horizontal flight distance of 125 feet from a launch angle of 30° above the horizontal. We may assume $x(0) = 0, y(0) = h$, and we find
$y'(0) = v_0\sin\dfrac{\pi}{6} = \dfrac{v_0}{2}$
$x'(0) = v_0\cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}v_0$
$y''(t) \equiv -32,\ x''(t) \equiv 0$
$y'(t) = -32t + \dfrac{v_0}{2},\ x'(t) = \dfrac{\sqrt{3}}{2}v_0$
$y(t) = -16t^2 + \dfrac{v_0}{2}t + h,$
$x(t) = \dfrac{\sqrt{3}}{2}v_0 t.$

$x(t)$ will be 125 if $t = 250/\left(\sqrt{3}\nu_0\right)$ at which time we require that y be h. Therefore
$-16\left(\dfrac{250}{\sqrt{3}v_0}\right)^2 + \dfrac{v_0}{2}\left(\dfrac{250}{\sqrt{3}v_0}\right) = 0$
$v_0 = \sqrt{\dfrac{8000}{\sqrt{3}}} \approx 68\text{ft/s}$

(b) With an angle of $45^\circ = \pi/4$, the equations become
$y'(0) = v_0\sin\dfrac{\pi}{4} = \dfrac{v_0}{\sqrt{2}}$
$x'(0) = v_0\cos\dfrac{\pi}{4} = \dfrac{v_0}{\sqrt{2}}$
$y''(t) = -32, \quad x''(t) = 0$
$y'(t) = -32t + \dfrac{v_0}{\sqrt{2}}, \quad x'(t) = \dfrac{v_0}{\sqrt{2}}$
$y(t) = -16t^2 + \dfrac{v_0 t}{\sqrt{2}} + h,$
$x(t) = \dfrac{v_0 t}{\sqrt{2}}$
where h is the height of the ramp.

We now solve $x(t) = 125$ which gives
$t_0 = t = \dfrac{125\sqrt{2}}{v_0}$
At this distance, we want the car to be at a height h to clear the cars. This gives the equation $y(t_0) = h$, or
$-16\left(\dfrac{125\sqrt{2}}{v_0}\right)^2 + \dfrac{125v_0\sqrt{2}}{v_0\sqrt{2}} + h = h$
Solving for v_0 gives
$v_0 = 20\sqrt{10} \approx 63.24$ ft/s.

26. Let $(x(t), y(t))$ be the trajectory. In this case,
$y(0) = 256, x(0) = 0$
$y'(0) = 0, x'(0) = 100$
$y''(t) \equiv 32,\ x''(t) \equiv 0$
$y'(t) = -32t, y(t) = -16t^2 + 256$
$x'(t) = 100, x(t) = 100t$

y will be zero when $t = 4$, at which time x will be 400. This is the drift distance.

27. (a) In this case with
$\theta_0 = 0$ and $\omega = 1$
$x''(t) = -25\sin(4t)$
$x'(0) = x(0) = 0$
$x'(t) = \dfrac{25}{4}\cos 4t - \dfrac{25}{4}$

$x(t) = \frac{25}{16}\sin 4t - \frac{25}{4}t$

(b) With $\theta_0 = \frac{\pi}{2}$ and $\omega = 1$

$x''(t) = -25\sin\left(4t + \frac{\pi}{2}\right)$

$x'(0) = x(0) = 0$

$x'(t) = \frac{25}{4}\cos\left(4t + \frac{\pi}{2}\right)$

$x(t) = \frac{25}{16}\sin\left(4t + \frac{\pi}{2}\right) - \frac{25}{16}$

28. (a) With $\theta_0 = \frac{\pi}{4}$ and $\omega = 2$

$x''(t) = -25\sin\left(8t + \frac{\pi}{4}\right)$

$x'(0) = 0 = x(0)$

$x'(t) = \frac{25}{8}\cos\left(8t + \frac{\pi}{4}\right) - \frac{25\sqrt{2}}{16}$

$x(t) = \frac{25}{64}\sin\left(8t + \frac{\pi}{4}\right) - \frac{25\sqrt{2}}{16}t - \frac{25\sqrt{2}}{128}$

(b) With $\theta_0 = \frac{\pi}{4}$ and $\omega = 1$

$x''(t) = -25\sin(4t + \pi/4)$

$x'(0) = x(0) = 0$

$x'(t) = \frac{25}{4}\cos(4t + \pi/4) - \frac{25\sqrt{2}}{8}$

$x(t) = \frac{25}{16}\sin(4t + \pi/4)\frac{25t\sqrt{2}}{8} - \frac{25\sqrt{2}}{32}$

29. The initial conditions are
$s(0) = 0, s'(0) = 0$.

Integrating $s''(t) = -32$ gives
$s'(t) = -32t + c_1$.
The initial condition gives
$s'(t) = -32t$.
Integrating gives
$s(t) = -16t^2 + c_2$.
The initial condition gives
$s(t) = -16t^2$.
Realizing that -32 was given in feet per second^2,and we are using centimeters now, we use, 1 foot $= 30.48$ cms
and get
$s(t) = -487.68t^2$ cm

The yardstick is grabbed when $s(t_0) = -d$, that is when
$t_0 = \frac{\sqrt{d}}{487.68} \approx 0.045\sqrt{d}$

30. Using the result from Exercise 15,
$v_1 = 8\sqrt{H}$.

Now we need to compute how big v_2 is in order for the ball to rebound to cH.

The initial conditions are
$v(0) = v_2, s(0) = 0$.
Integrating $a(t) = -32$ gives
$v(t) = -16t + v(0) = -16t + v_2$
Integrating again we get
$s(t) = -8t^2 + v_2 t + s(0) = -8t^2 + v_2 t$
$s(t_0) = cH$ when $v(t_0) = 0$, that is when $t_0 = v_2/16$

$-8\left(\frac{v_2}{16}\right)^2 + v_2\left(\frac{v_2}{16}\right) = cH$

$\frac{v_2^2}{32} = cH$

$v_2 = \sqrt{32cH}$

Now the coefficient of restitution is
$\frac{v_2}{v_1} = \frac{\sqrt{32cH}}{8\sqrt{H}} = \sqrt{\frac{c}{2}}$

31. From Exercise 5, time of impact is
$t = \frac{\sqrt{30}}{4}$ seconds.
$2\frac{1}{2}$ somersaults corresponds to 5π radians of revolution.
Therefore the average angular velocity is
$\frac{5\pi}{\sqrt{30}/4} = \frac{20\pi}{\sqrt{30}} \approx 11.47$ rad/sec

32. The initial conditions are
$y(0) = 10$, $y'(0) = 160\sin 45°$
$x(0) = 0$, and $x'(0) = 160\cos 45°$

Integrating $x''(t) = 0$ and $y''(t) = -32$ and using the initial conditions gives
$x'(t) = 80\sqrt{2}$
$x(t) = (80\sqrt{2})t$
$y'(t) = -32t + 80\sqrt{2}$
$y(t) = -16t^2 + (80\sqrt{2})t + 10$.

We now want to solve for when $y(t) = 5$, which gives the equation
$-16t^2 + (80\sqrt{2})t + 10 = 5$
Solving gives
$t = \frac{-80\sqrt{2} \pm \sqrt{12800 + 640}}{-32} \approx -0.087, 7.16$.
We, of course, take the positive solution.
$x(7.16) = (80\sqrt{2})(7.16) \approx 810.1$.
So, place the net 810.1 feet away from the cannon.
$y'(7.16) = -32(7.16) + 80\sqrt{2} \approx 116.0$
Since we have $x' = 80\sqrt{2} \approx 113.1$, this means that the impact velocity is
$v = \sqrt{(x')^2 + (y')^2}$
$= \sqrt{(116.0)^2 + (113.1)^2} \approx 162.0$

which means the Flying Zucchini comes down squash.(We should have known this—the velocity at a height of 10 should have been equal to his initial velocity so his velocity at a height of of 5 should be slightly higher, which it is.)

33. Let $(x(t), y(t))$ be the trajectory of the center of the basketball. We are assuming that $y(0) = 6$, $x(0) = 0$, the angle of launch θ of the shot is $52°$ ($\theta = \dfrac{13\pi}{45}$ in radians) and the initial speed is 25 feet per second. Therefore

$y'(0) = 25 \sin \dfrac{13\pi}{45} \approx 19.70$

$x'(0) = 25 \cos \dfrac{13\pi}{45} \approx 15.39$

$y''(t) \equiv -32, x''(t) \equiv 0$
$y'(t) = -32t + 19.70, x'(t) \equiv 15.39$
$y(t) = -16t^2 + 19.70t + 6,$
$x(t) = 15.39t.$
x will be 15 when t is about
$15/15.39 = .9746\ldots$, at which time y will be about
$-16(.9746\ldots)^2 + 19.70(.9746\ldots) + 6 \approx 10$

In other words, the center of the ball is at position $(15, 10)$ and the shot is good. More generally, with unknown θ, the number 19.70 is replaced by $25 \sin \theta$, while the number 15.39 is replaced by $25 \cos \theta$. y will be exactly 10 if

$-16t^2 + 25t \sin \theta + 6 = 10$

$t = \dfrac{25 \sin \theta + \sqrt{625 \sin^2 \theta - 256}}{32}$

$x = 25t \cos \theta.$

As a function of θ, this last expression is too complicated to use calculus (easily) to maximize and minimize it on the θ-interval $(48°, 57°)$, but quick spreadsheet calculations give these values:
(Observe that x is not a monotonic function of θ in this range. It takes its maximum when θ is between 52.4 and 52.5 degrees. The evidence is overwhelming that all the shots will be good.)

θ	t	x
degrees	seconds	feet
48.0	0.8757	14.6484
49.0	0.9021	14.7958
50.0	0.9274	14.9024
51.0	0.9516	14.9710
52.0	0.9748	15.0038
52.1	0.9771	15.0051
52.2	0.9793	15.0062
52.3	0.9816	15.0069
52.4	0.9838	15.0073
52.5	0.9861	15.0073
52.6	0.9883	15.0070
52.7	0.9905	15.0064
52.8	0.9928	15.0054
52.9	0.9950	15.0042
53.0	0.9972	15.0026
54.0	1.0187	14.9690
55.0	1.0394	14.9044
56.0	1.0594	14.8100
57.0	1.0787	14.6869

34. Let$(x(t), y(t))$ be the trajectory of the centre of the basketball.

Here $y(0) = 8, x(0) = 0, \theta = 30^0$ and $v = 27$. Therefore y'(0)=27sin$\dfrac{\pi}{6} = 13.5$ and
$x'(0) = 27 \cos \dfrac{\pi}{6} = 23.3827$
$y''(t) \equiv -32 \Rightarrow y'(t) = -32t + 13.5,$
Or $y(t) = -16t^2 + 13.5t + 8$ also,
$x''(t) \equiv 0 \Rightarrow x'(t) \equiv 23.3827$

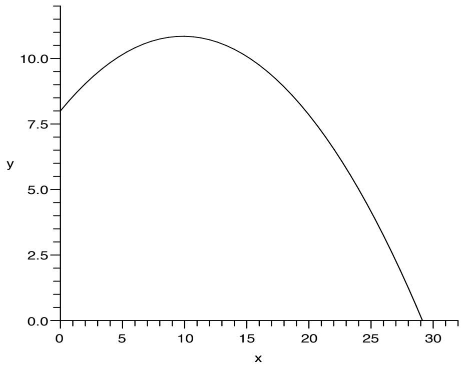
That is $x(t) = (23.3827)\, t$

(a) Consider $x(t) = 15$
$\Rightarrow t = \dfrac{15}{23.3827} \approx 0.6415,$
for which
$y(0.6415)$
$= -16(0.6415)^2 + 13.50\,(0.6415) + 8$
$= 10.0759$

Now, $y(t) = 10 \Rightarrow t \approx 0.6520$ for which

$x(0.6520) = (23.3827)\,(0.6520)$
≈ 15.2455

It is evident from the above calculations that the centre of the ball passes through $(15, 10.0759)$ and $(15.2455, 10)$. This means that the centre of the ball goes through the basket. The graph of the motion is as follows



(b) When $x(t) = 14.25 \Rightarrow t \approx 0.6094$ this gives $y(t) = 10.2849$.
That is $(14.25, 10.2845)$ lies on the curve. Therefore the minimum distance between the centre of the ball and the front rim is 0.2845. The minimum distance between the centre of the ball and the back rim at $(15.75, 10)$ is $0.5045'$.

(c) If the ball is of diameter, then its radius is . Since the minimum distance between the center of the ball and the front rim is less than the radius of the ball, the ball hits the front rim.

35. (a) $85° = \frac{17}{36}\pi$ radiance.
$x'(0) = 100 \cdot \cos\left(\frac{17}{36}\pi\right) \approx 8.72$
$y'(0) = 100 \cdot \sin\left(\frac{17}{36}\pi\right) \approx 99.62$
$x''(0) = -20$
$y''(0) = 0$
$y(t) = 99.62t$
$x(t) = -10t^2 + 8.72t$
$y(t_0) = 90$ when $t_0 = 0.903$
$x(t_0) = x(0.903) \approx -0.29$
The ball just barely gets into the goal.

(b) Use the calculation from Exercise 35.(a),
$y(t_1) = 10$ when $t_1 = 0.100$
$x(t_1) = x(0.100) \approx 0.775$
The kick does not go around the wall.

36. Let $(x(t), y(t))$ be the trajectory of the ship. Some of our data is in feet, so we will take $g = -32$ in this problem. We have
$y''(t) = 32$
$y'(t) = -32t + y'(0)$
$y(t) = -16t^2 + y'(0)t + y(0)$
$x'(t) \equiv c$
$x(t) = ct + x(0)$

Solving for t, we have
$\frac{1}{c}(x - x(0)) = t.$
Substituting this expression for t in $y(t)$, we have
$y - y(0)$
$= -16\left[\frac{1}{c}(x - x(0))\right]^2 + y'(0)\left[\frac{1}{c}(x - x(0))\right]$
Hence the path is a parabola.
Turning to the question of the duration of weightlessness, we can assume $x(0) = 0$, and we know that $y'(t) = 0$ when $y - y(0) = 2500$. For this unknown time t_1 (the moment when y' is zero), we have $0 = -32t_1 + y'(0)$.
Therefore $t_1 = y'(0)/32$, and
$2500 = y(t_1) - y(0)$
$= -16\left[\frac{y'(0)}{32}\right]^2 + y'(0)\left[\frac{y'(0)}{32}\right]$
$= \frac{y'(0)^2}{64}$,
hence $y'(0)2 = 64(2500)$
$y'(0) = 8(50) = 400$, and
$t_1 = 400/32 = 25/2$.

We now know that $y - y(0) = -16t^2 + 400t$ for all t.

The second time (t_2) that $y(t) = y(0)$ (after time zero) occurs when $t = 400/16 = 25$ seconds.
This is the duration of the weightless experience. Note that $t_2 = 2t_1$. The plane must pull out of the dive soon after this time.

37. Let $y(t)$ be the height of the first ball at time t, and let v_{0y} be the initial velocity. We can assume $y(0) = 0$. As usual, we have
$y'' = -32$, $y' = -32t + v_{0y}$,
$y = -16t2 + tv_{0y}$.

The second return to height zero is at time $t = 16/v_{0y}$. If this is to be $5/2$, then $v_{0y} = 40$.
But the maximum occurs at time
$v_{0y}/32 = 5/4$
at which time the height $(y(5/4))$ is $-16(25/16) + 40(5/4) = 25$feet.
For eleven balls, the difference is that the second return to zero is to be at time $11/4$, hence $v_{0y} = 44$, and the maximum height is 30.25.

38. In this case, we start with initial conditions
$x'(0) = v_{0x}, x(0) = 0; y'(0) = v_{0y}, y(0) = 0.$
Integrating $x''(t) = 0$ and $y''(t) = -32$ and using the initial conditions gives
$x'(t) = v_{0x}$
$x(t) = v_{0x}t$
$y'(t) = -32t + v_{0y}$
$y(t) = -16t + v_{0y}t$
The ball is caught when $y(t) = 0$ so we solve

this equation to get $t = \frac{v_{0y}}{16}$. Plugging this into $x(t)$ gives the horizontal distance
$\omega = x\left(\frac{v_{0y}}{16}\right) = \frac{v_{0x}v_{0y}}{16}$.

39. The student must first study the solution to Exercise 38. Here we have the additional x-component of the motion, which as in so many problems is $x(t) = tv_{0x}$. With initial *speed* of v_0, and initial angle α from the *vertical,* we have $v_{0y} = v_0 \cos\alpha$ and
$v_{0x} = v_0 \sin\alpha$.The horizontal distance at elapsed time $v_{0y}/16$ (time of return to initial height) is by formula
$x(v_{0y}/16) = (v_{0y}/16)v_{0x}$ which defines ω.
As in Exercise 37, the maximum height occurs at time $v_{0y}/32$, and at this time the height h is
$-16(v_{0y}/32)^2 + v_{0y}(v_{0y}/32) = v_{0y}^2/64$
$= (v0y/64)(16\omega/v_{0x})$
$= (\omega/4)(\cos\alpha/\sin\alpha) = \omega/(4\tan\alpha)$.

Thus $\omega = 4h\tan\alpha$.

40. The linear approximation is $\tan^{-1} x = x$, i.e., $\tan x \approx x$ From Exercise 43, we have
$\omega = 4h\tan\alpha$

Applying the linearization gives
$\omega = 4h\tan\alpha \approx 4h\alpha$
or $\alpha \approx \frac{\omega}{4h}$
This shows that $\Delta\alpha \approx \frac{\Delta\omega}{4h}$

41. We must use the result
$\Delta\alpha \approx \frac{\Delta\omega}{4h}$ from Exercise 40.

With $h = 25$ from Exercise 51 (10 balls) and $\omega = 1$, we get
$\Delta\alpha$ about $1/100 = .01$ radians
or about $.6°$

42. In this case, the height to juggle 11 balls is 30.25 feet. Therefore with $\Delta\omega = 1$, we get
$\Delta\alpha \approx \frac{\Delta\omega}{4h} = \frac{1}{4(30.25)} \approx 0.0083$ rad or about $0.47°$.

43. With trajectory (x, y), and assuming
$x(0) = 0$ and $y(0) = 0$, we have by now seen many times the conclusion $y = -gt^2 + tv\sin\theta$.

The return to ground level occurs at time $t = 2v\sin\theta/g$, at which time the horizontal range is $x = tv\cos\theta = v^2\sin(2\theta)/g$.

With $v = 60$ ft per second and $\theta = 25°$, and on earth with $g = 32$, this is about 86 feet, a short chip shot. On the moon with $g = 5.2$, it is about 530.34 ft.

44. Let $((x(t), y(t))$ be the trajectory of the initial burst of water. If the angle of inclination of the hose is θ, we have the relations
$\tan\theta = m$
$\sin\theta = \frac{m}{\sqrt{1+m^2}}$
$\cos\theta = \frac{1}{\sqrt{1+m^2}}$
We assume $x(0)$ =0 and $y(0) = 0$ and then find
$y''(t) \equiv -32$
$y'(t) = -32t + v\sin\theta$
$y = y(t) = -16t^2 + tv\sin\theta$
$y = y(t) = -\,16t^2 + \frac{tvm}{\sqrt{1+m^2}}$
$x'(t) \equiv v\cos\theta$
$x = x(t) = tv\cos\theta = \frac{tv}{\sqrt{1+m^2}}$
Solving the last equation in the form
$t = \frac{x\sqrt{1+m^2}}{v}$
and inserting this in the y-formula, we find
$y = -16x^2\frac{(1+m^2)}{v^2} + mx$.

45. Let $(x(t), y(t))$ be the trajectory of the paint ball, and let $z(t)$ be the height of the target at time t. We do assume that
$y(0) = z(0)$ (target opposite shooter at timeof shot) and
$y'(0) = 0$ (aiming directly at the target, hence using an initially horizontal trajectory), and as a result $y - z$ has second derivative 0, and initial value 0.

However, this only tells us that
$y - z = [y'(0) - z'(0)]t = -z'(0)t$
and if the target is already in motion ($z'(0)$ not zero), the shot may miss at 20 feet or any distance.

If on the other hand, the target is stationary at the moment of the shot, then the shot hits at20 feet or any other distance.

46. In this problem, we have the falling object with initial conditions
$y_1'(0) = 0, y_1(0) = 100$.

The object that is launched from the ground has initial conditions
$y_2'(0) = 40, y_2(0) = 0$

We now integrate the equations
$y_1''(t) = -32$ and $y_2''(t) = -32$, using the initial conditions, to get

$y_1'(t) = -32t$
$y_1(t) = -16t^2 + 100$
$y_2'(t) = -32t + 40$
$y_2(t) = -16t^2 + 40t$
Now, we just solve $y_1(t) = y_2(t)$, or
$-16t^2 + 100 = -16t^2 + 40t$
Solving gives $t = 2.5$, so the objects collide after 2.5 seconds and this collision occurs at a height of $y_1(2.5) = 0$.

This may seem odd, but notice that the maximum height of the y_2 object is only 25 feet. What this means is that the y_2 object goes up and then down and then the y_1 object only catches the y_1 object when both objects actually hit the ground!

47. (a) The speed at the bottom is given by
$\frac{1}{2}mv^2 = mgH, v = \sqrt{2gH}$

(b) Use the result from (a)
$v = \sqrt{2gH} = \sqrt{2 \cdot 16g} = 4\sqrt{2g}$
$= 4\sqrt{2 \cdot 32} = 32\text{ft/s}$

(c) At half way down,
$\frac{1}{2}mv^2 + mh8 = mh16,$
$v = \sqrt{2 \cdot (16-8)g} = 4\sqrt{g}$
$= 4\sqrt{32} \approx 22.63\text{ft/s}$

(d) At half way down, the slope of the line tangent to $y = x^2$ is, $2 \cdot \sqrt{8} = 4\sqrt{2}$
Hence we know that
$\frac{v_y}{v_x} = 4\sqrt{2}$
At the same time,
$(v_y)^2 + (v_x)^2 = (4\sqrt{g})^2$
$v_x^2 = \frac{16g}{33}$
$v_x = 4\sqrt{\frac{g}{33}} \approx 3.939 \text{ ft/s}$
$v_y = 16\sqrt{\frac{2g}{33}} \approx 22.282 \text{ ft/s}$

48. First we compute the speed v of the bowling ball at the moment when it rolls right out of the window.

$30 = 16t_0^2, t_0 = \frac{\sqrt{30}}{4}$
$10 = t_0 v_0, v_0 = \frac{40}{\sqrt{30}}.$
From conservation of energy
$\frac{1}{2}mv^2 = mgh,$
$\frac{1}{2}m\left(\frac{40}{\sqrt{30}}\right)^2 = mgh$
$\frac{80}{3} = 32 \cdot h,$
$h = \frac{5}{6}$
The height of the ramp should be $\frac{5}{6}$.

5.6 Applications of Integration to Physics and Engineering

1. We first determine the value of the spring constant k. We convert to feet so that our units of work is in foot-pounds.
$5 = F(1/3) = \frac{k}{3}$ and so $k = 15$.
$W = \int_0^6 F(x)dx$
$= \int_0^{1/2} 15x dx = \frac{15}{8}$ foot-pounds.

2. We first determine the value of the spring constant k. We convert to feet so that our units of work is in foot-pounds.
$10 = F(1/6) = \frac{k}{6}$ and so $k = 60$.
$W = \int_0^3 F(x)dx$
$= \int_0^{1/4} 60x dx = \frac{15}{8}$ foot-pounds.

3. The force is constant (250 pounds) and the distance is 20/12 feet, so the work is
$W = Fd = (250)(20/12)$
$= 1250/3$foot-pounds.

4. The force is constant (300 pounds) and the distance is 6 feet, so the work is
$W = Fd = (300)(6) = 1800$ foot-pounds.

5. If x is between 0 and $30,000$ feet, then the weight of the rocket at altitude x is
$10000 - \frac{1}{15}x.$
Therefore the work is
$\int_0^{30{,}000} \left(10{,}000 - \frac{x}{15}\right)dx$
$= \left(10{,}000x - \frac{x^2}{30}\right)\Big|_0^{30{,}000}$
$= 270{,}000{,}000$ ft-lb

6. If x is between 0 and $10,000$ feet, then the weight of the rocket at altitude x is $8000 - \frac{x}{10}$. Therefore the work done is

$$W = \int_0^{10,000} \left(8000 - \frac{x}{10}\right) dx$$
$$= 60,800,000\text{ft-lb}$$

7. The weight of the 40 feet long chain is 1000 pounds. Therefore the weight of the 30 feet long chain is 750 pounds. The force acting here is 750 pounds and the distance traced due to the applied force is 30 feet. Hence the work done is

$$W = Fd$$
$$= (750) \cdot (30)$$
$$= 22500\,\text{foot-pounds.}$$

8. Let x be the distance of the bucket from the initial position. Consequently x increases from 0 to 80. As the sand from the bucket leaks at rate of 2 lb/s, the weight of bucket at the distance x is $\left(100 - \frac{x}{2}\right)$. Therefore work done is

$$W = \int_0^{80} \left(100 - \frac{x}{2}\right) dx = \left(100x - \frac{x^2}{4}\right)_0^{80}$$
$$= 8000 - 1600$$
$$= 6400\,\text{ft-lb.}$$

9. (a) $W = \int_0^1 800x(10x)dx$

$$= \left(400x^2 - \frac{800}{3}x^3\right)\Big|_0^1$$
$$= \frac{400}{3}\text{ mile-lb}$$
$$= 704{,}000\text{ ft-lb}$$

(b) Horsepower is not equal to $800x(1-x)$ because this is the derivative with respect to distance and not with respect to time. Average horsepower is the ratio of total work done divided by time: $\dfrac{704,000\text{ ft-lb}}{80\text{ s}} = 16\text{ hp}$

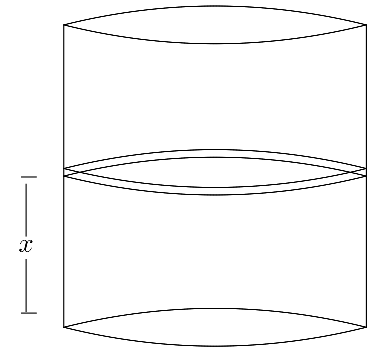
10. (a) $W = \int_0^{100} 62.4\pi(100x - x^2)(200 + x)dx$

$$= 62.4\pi \int_0^{100} \left(20{,}000x - 100x^2 - x^3\right) dx$$
$$= 8{,}168{,}140{,}899\text{ ft-lb}$$

(b) This is the same as Exercise 10.(a) except the limits of integration change to reflect that the tank is only filled half way:

$$W = \int_0^{50} 62.4\pi(100x - x^2)(200 + x)dx$$
$$= 3,777,765,166\text{ ft-lb}$$

11. (a)



Let x represent the distance measured (in ft) from the bottom of the tank, as shown in the above diagram. The entire tank corresponds to the interval
$0 \le x \le 9.843$ (1 mt = 3.281 ft).
Let us partition the tank into
$0 = x_0 < x_1 < x_2 < ... < x_n = 9.843.$

such that
$$x_i - x_{i-1} = \Delta x = \frac{9.843}{n}$$
for each $i = 1, 2, 3, , n.$

This partitions the tank into n layers, each corresponding to an interval $[x_{i-1}, x_i]$.

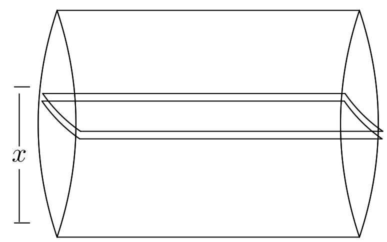
Let us consider a water layer corresponding to $[x_{i-1}, x_i]$, which is a cylinder of height Δx and radius 3.281 ft(1mt) . This layer must be pumped at a distance of $(9.843 - c_i)$ for $c_i \in [x_{i-1}, x_i]$

Thus the force exerted in doing so,is
$F_i \approx$ (Volume of the cylindrical slice)
$\times$ (Weight of the water per unit volume)
$\approx \pi(3.281)^2\,(\Delta x) \times (62.4)$
$\approx 2110.31\,(\Delta x)$

Thus the corresponding work done
$W_i = 2110.31\,(9.843 - c_i)\,(\Delta x)$

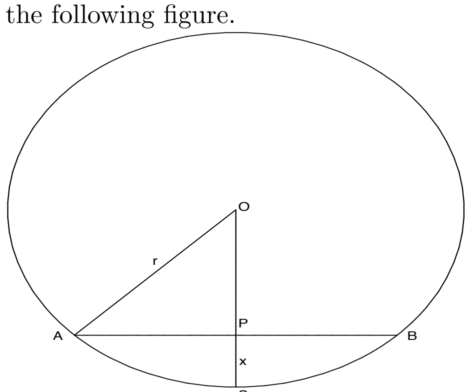
Therefore the total work done
$$W = \lim_{n\to\infty} \sum_{i=1}^{n} (2110.31\,(9.843 - c_i)\,(\Delta x))$$
$$= 2110.31 \int_0^{9.843} (9.843 - x)dx$$
$$= 2110.31 \left(9.843x - \frac{x^2}{2}\right)\Big|_0^{9.843}$$
$$= 102228.48\text{ feet pounds}$$

(b)



Let x represent the distance measured (in ft) from the bottom of the tank, as shown in the above diagram. The entire tank corresponds to the interval $0 \le x \le 3.281$ (as 1mt = 3.281 ft). Let us partition the tank into
$0 = x_0 < x_1 < x_2 < ... < x_n = 3.281.$ such that
$x_i - x_{i-1} = \Delta x = \dfrac{3.281}{n}$ for each i =1,2,3,,n. This partitions the tank into n layers, each corresponding to an interval $[x_{i-1}, x_i]$. Let us consider a water layer corresponding to $[x_{i-1}, x_i]$. Which is a cuboid of length 9.843, width $2\sqrt{6.562x - x^2}$ and height Δx.

The width is calculated with the help of the following figure.



In the above figure O is the centre of the circle of radius r. $OP = r - x$,
$AP = \sqrt{r^2 - (r-x)^2} = \sqrt{2rx - x^2};$
$AB = 2\sqrt{2rx - x^2}$

The said layer must be pumped at a distance of $(2r - c_i)$ for $c_i \in [x_{i-1}, x_i]$. Thus the force exerted in doing so, is
$F_i \approx$ (Volume of the cuboid shaped slice)
$\times$ (Weight of the water per unit volume)
= (length × width × height) × (62.4)
$\approx \left(9.843 \times 2\sqrt{6.562x - x^2} \times \Delta x\right) \times$ (62.4)
$\approx 1228.41\sqrt{6.562x - x^2}\,(\Delta x)$
Thus the corresponding work done
$W_i = 1228.41\sqrt{6.562x - x^2}\,(6.562 - c_i)\,(\Delta x)$
Therefore the total work done

$$W = (1228.41) \times \lim_{n\to\infty} \sum_{i=1}^{n} \left(\sqrt{6.562x - x^2}\,(6.562 - c_i)\,\Delta x\right)$$
$$= 1228.41 \int_0^{6.562} \sqrt{6.562x - x^2}\,(6.562 - x)dx$$
$$= 136304.64 \text{ feet pounds}$$

12. We set up our coordinates similar to Example 6.3, with x representing vertical distance from the vertex (the bottom of the tank). If slice the water in horizontal slices, these slices have radius $r = \dfrac{x}{2}$ and the volume of a cylindrical slice is $\pi r^2 \Delta x = \dfrac{\pi x^2}{4}\Delta x$. The weight density of water is 62.4, which gives the force exerted by this slice of water as $15.6\pi x^2 \Delta x$. This slice of water must travel up a distance of $10 - x$ and therefore the work required to pump this slice out of the tank is

$W_i \approx 15.6\pi x^2 \Delta x(10 - x)$

$\approx 15.6(10 - x)\pi x^2 \Delta x$

Now, we add up the work for all the slices and turn it into an integral.

$$W = \int_0^{10} 15.6(10 - x)\pi x^2 dx$$
$$= 15.6\pi \left(\frac{2500}{3}\right)$$
$$\approx 40841 \text{ foot-pounds}$$

13. $W = \displaystyle\int_0^{10} axdx = \frac{100a}{2}$

$W_1 = \displaystyle\int_0^{c} axdx = \frac{ac^2}{2}$

$W_1 = \dfrac{W}{2}$ gives $\dfrac{ac^2}{2} = \dfrac{1}{2}\dfrac{100a}{2}$

$c = \sqrt{50} \approx 7.1$ feet

The answer is greater than 5 feet because the deeper the laborer digs, the more distance it is required for him to lift the dirt out of the hole.

14. By calculation, the width at x feet depth is $5 - x/2$, therefore

$$W(x) = \int_0^{x} t\left(5 - \frac{t}{2}\right) dt = v52x^2 - \frac{1}{9}x^3$$

$W(6) = 66$
Solving $\frac{5}{2}x^2 - \frac{1}{9}x^3 = 33$ we get
$x \approx 4.0$ feet

15. We estimate the integral using Simpson's Rule:
$J = \int_0^{.0008} F(t)dt$
$\approx \frac{.0008}{3(8)}[0 + 4(1000) + 2(2100)$
$+ 4(4000) + 2(5000) + 4(5200)$
$+ 2(2500) + 4(1000) + 0]$
≈ 2.133
$2.13 = J = m\Delta v = .01\Delta v$
$\Delta v = 213$ ft/sec
The velocity after impact is therefore
$213 - 100 = 113$ ft/sec.

16. We compute the impulse using Simpson's rule:
$J \approx \frac{.6}{3(6)}[0 + 4(8000) + 2(16,000)$
$+4(24,000) + 2(15,000) + 4(9000)[5pt] \quad +0]$
≈ 7533.3
$7533.3 = J = m\Delta v = 200\Delta v$
$\Delta v = 37.7$ ft/sec

Since the velocity after the crash is zero, this number is the estimated original velocity.

17. $F'(t)$ is zero at $t = 3$, and the maximum thrust is $F(3) = 30/e \approx 11.0364$

It is implicit in the drawing that the thrust is zero after time 6. Therefore the impulse is $\int_0^6 10te^{-t/3}dt = 90 - 270e^{-2} \approx 53.55$.

18. The impulse is
$J = \int_0^6 F(t)\,dt = 48$. The impulse of Exercise 17 was about 53.55 which means that the rocket of Exercise 17 would have greater velocity and therefore a higher altitude.

19. $m = \int_0^6 \left(\frac{x}{6} + 2\right) dx = 15$
$M = \int_0^6 x\left(\frac{x}{6} + 2\right) dx = 48$
Therefore,
$\bar{x} = \frac{M}{m} = \frac{48}{15} = \frac{16}{5} = 3.2$
So the center of mass is to the right of $x = 3$.

20. $m = \int_0^6 \left(3 - \frac{x}{6}\right) dx = 15$
$M = \int_0^6 x\left(3 - \frac{x}{6}\right) dx = 42$
So, therefore
$\overline{x} = \frac{M}{m} = \frac{42}{15} = \frac{14}{5} = 2.8$
So the center of mass is to the left of $x = 3$.

21. $m = \int_{-3}^{27} \left(\frac{1}{46} + \frac{x+3}{690}\right)^2 dx$

$= \frac{690}{3}\left(\frac{1}{46} + \frac{x+3}{690}\right)^3 \Big|_{-3}^{27}$

$\approx .0614$ slugs ≈ 31.5 oz

22. $m = \int_0^{32} \left(\frac{1}{46} + \frac{x+3}{690}\right)^2 dx$

≈ 0.08343 slugs ≈ 42.418 oz

23. $M = \int_{-3}^{27} x\left(\frac{1}{46} + \frac{x+3}{690}\right)^2 dx$

≈ 1.0208
$\bar{x} = \frac{M}{m} = \frac{1.0208}{.0614} \approx 16.6$ in.
This is 3 inches less than the bat of Example 6.5, a reflection of the translation three inches to the left on the number line.

24. $M = \int_0^{32} x\left(\frac{1}{46} + \frac{x+3}{690}\right)^2 dx$
≈ 1.72495

$\bar{x} = \frac{M}{m} = 20.6745$
Compared to the baseball bat of Example 6.5, this baseball bat is longer and therefore has more mass further out.

25. $m = \int_0^{30} .00468\left(\frac{3}{16} + \frac{x}{60}\right) dx$
$\approx .0614$ slugs
$M = \int_0^{30} .00468x\left(\frac{3}{16} + \frac{x}{60}\right) dx$
≈ 1.0969
weight $= m(32)(16) = 31.4$ oz

$\bar{x} = \frac{M}{m} = \frac{1.0969}{.0614} \approx 17.8$ in.

26. The center of mass of the wooden bat of Example 6.5 is at 19.6 inches. The center of mass of the aluminum bat of Exercise 25 is at 17.8 inches—moving the sweet spot to the inside.

27. Area of the base is $\frac{1}{2}\,(3 + 1) = 2$.

Area of the body is $1 \times 4 = 4$.

Area of the tip is $\frac{1}{2}(1 \times 1) = \frac{1}{2}$.

Base:
$m = \int_0^1 \rho(3-2x)dx = \frac{5}{12} \approx .4167.$
Body:
$m = \int_1^5 \rho dx = 12\rho$
$\bar{x} = \frac{M}{m} = 3$
Tip:
$m = \int_5^6 \rho(6-x)dx \approx 2.67\rho$
$\bar{x} = \frac{M}{m} = \frac{16}{3} \approx 5.33$

28. We use the coordinate system as in Exercise 29, with $x = 0$ corresponding to the left of the rocket.

From Exercise 27, the base has total mass $\frac{5}{6}\rho$ and center of mass at $x = \frac{5}{12}$.
From Exercise 27, the body has total mass 12ρ and center of mass at $x = 3$.
From Exercise 27, the tip has total mass $\frac{1}{2}\rho$ and center of mass at $x = \frac{16}{3}$.
The total mass of these three particles is $m = \frac{40}{3}\rho$ and the moment of these particles is

$$M = \left(\frac{5}{6}\rho\right)\left(\frac{5}{12}\right) + (12\rho)(3) + \left(\frac{1}{2}\rho\right)\left(\frac{16}{3}\right) = \frac{2809}{72}\rho$$

The center of mass of the system is

$$\overline{x} = \frac{M}{m} = \left(\frac{2809}{72}\rho\right)\left(\frac{3}{40\rho}\right) = \frac{2809}{960} \approx 2.926$$

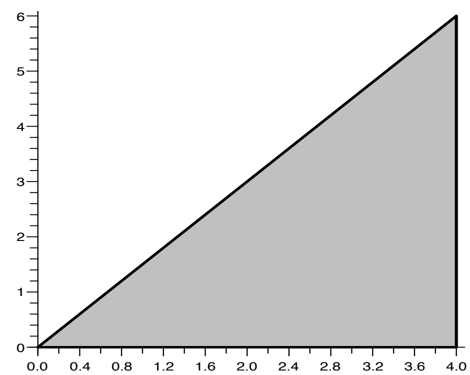
29. The x-coordinate of the centroid is the same as the center of mass from $x = 0$ to $x = 4$ with density $\rho(x) = \frac{3}{2}x$, hence

$$\bar{x} = \frac{M}{m} = \frac{\int_0^4 3/2 \cdot x^2 dx}{\int_0^4 3/2 \cdot x dx} = \frac{8}{3}$$

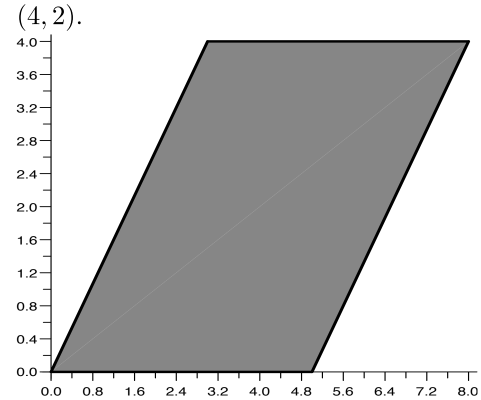
The y-coordinate of the centroid is the same as the center of mass from $y = 0$ to $y = 6$ with density $\rho(y) = 6 - \frac{2}{3}y$, hence

$$\bar{y} = \frac{M}{m} = \frac{\int_0^6 2/3 \cdot \left(6y - \frac{2}{3}y^2\right)\, dy}{\int_0^6 2/3 \cdot \left(6 - \frac{2}{3}y\right) dy} = 2$$

So the center of the given triangle is the point $(8/3, 2)$.



30. Again we need to find both the x-coordinate and y-coordinate of the centroid. But in this case, since everything is symmetric, in fact we can easily see that the centroid is going to be $(4, 2)$.

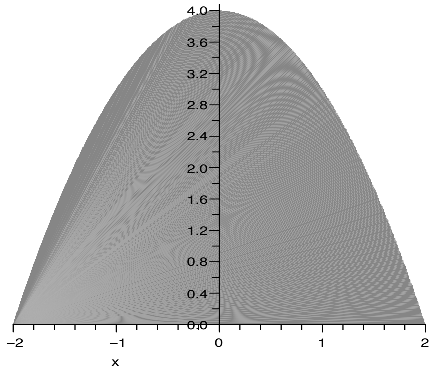


31. This time the x-coordinate of the centroid is obviously $x = 0$, so the question remains to find the y-coordinate.

This is the same as finding the center of mass from $y = 0$ to $y = 4$ with density $\rho(y) = \sqrt{4-y}$, hence

$$\bar{y} = \frac{M}{m} = \frac{\int_0^4 y\sqrt{4-y}\, dy}{\int_0^4 \sqrt{4-y} dy} = \frac{-\int_4^0 (4u^{1/2} - u^{3/2})\, du}{-\int_4^0 u^{1/2}\, du} = \frac{(8/3 \cdot u^{3/2} - 2/5 \cdot u^{5/2})\big|_0^4}{2/3 \cdot u^{3/2}\big|_0^4} = \frac{8}{5}$$

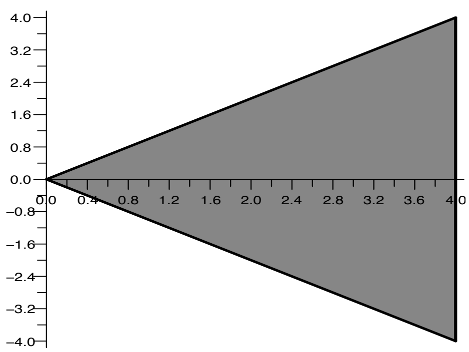
So the centroid is the pint $(0, 8/5)$.



32. This time the y-coordinate is obviously $y = 0$. The x-coordinate can found using the density $\rho(x) = 2x$, from $x = 0$ to $x = 4$, and

$$\bar{x} = \frac{M}{m} = \frac{\int_0^4 2x^2 dx}{\int_0^4 2x dx} = \frac{8}{3}$$

So the centroid is $(8/3, 0)$.



33. With x the depth, the horizontal width is a linear function of x, given by $x + 40$. Hence,

$$F = \int_0^{60} 62.4x(x + 40)dx$$

$$= 62.4 \left(\frac{x^3}{3} + 20x^2 \right) \Bigg|_0^{60} = 8{,}985{,}600 \text{ lb}$$

34. In this case, we just change the limits of integration.

$$F = \int_{10}^{60} 62.4x(x + 40)\, dx = 8,840,000 \text{ lb}$$

35. Let x be the vertical deviation above the center of the window, the horizontal width of the window is given by $2\sqrt{25 - x^2}$, depth of water $40 + x$, and hydrostatic force

$$62.4 \int_{-5}^{5} (x + 40) 2\sqrt{25 - x^2} dx$$

$$= 62.4 \int_{-5}^{5} 2x\sqrt{25 - x^2} dx$$

$$+ 62.4(40) \int_{-5}^{5} 2\sqrt{25 - x^2} dx$$

$$\approx 196,035 \text{ pounds.}$$

36. Let x be the distance from the surface of the water. For a given value of x, the width of the window is constant, 40. The force exerted on the window by a slice of water, of depth x is $F_i \approx (62.4)(40)x\Delta x$.

We sum these forces up over the height of the window and turn it into an integral:

$$F = \int_0^{10} (62.5)(10)x dx = 31,250 \text{ lb.}$$

37. Assuming that the center of the circular window descends to 1000 feet, then by the previous principle, after converting the three inch radius to 1/4 feet, we get $F = 12{,}252$ pounds. An alternate calculation in which x is the deviation downward from the top edge of the window, would be

$$F = \int_0^{0.5} 62.4(999.75 + x)$$

$$\cdot\, 2\sqrt{(0.25)^2 - (0.25 - x)^2} dx$$

$$= \int_0^{0.5} 124.8(999.75 + x)\sqrt{0.5x - x^2} dx$$

$$\approx 12,252 \text{ lb}$$

38. Due to the fact that the size of the watch is so small, we can assume that the force will be approximately the same regardless of orientation of the watch.
The hydrostatic force is given by $F = \rho dA$ where, ρ is the density of the water (62.4), d is the depth (60), and A is the area, $A = \pi(1/12)^2$.
Putting these together gives
$F \approx (62.4)(60)(\pi/144) \approx 81.68$ lb.

39. (100 tons)(20 miles/hr)

$$= \frac{(100 \cdot 2000 \text{ lbs})(20 \cdot 5280 \text{ ft})}{3600\text{sec}}$$

$$\approx 5{,}866{,}667 \text{ ft-lb/s}$$

$$= \frac{5{,}866{,}667}{550} \text{ hp}$$

$$\approx 10,667 \text{ hp}$$

40. This is a matter of slicing and approximating. Divide the subinterval $[a, b]$ into n equal subintervals. Then, we take the limit as $n \to \infty$, which turns the Riemann sum into an integral.

$$J \approx \sum_{i=1}^{n} F(t_i)\Delta t.$$

$$J = \lim_{n\to\infty}\sum_{i=1}^{n} F(t_i)\Delta t = \int_a^b F(t)\,dt$$

41. The bat in Exercise 23 models the bat of Example 6.5 choked up 3 in.

From Example 6.5:
$f(x) = \left(\frac{1}{46} + \frac{x}{690}\right)^2$;
$\int_{-3}^{27} f(x)\cdot x^2 dx \approx 27.22.$
From Exercise 23:
$f(x) = \left(\frac{1}{46} + \frac{x+3}{690}\right)^2$;
$\int_{-3}^{27} f(x)\cdot x^2 dx \approx 20.54.$
Reduction in moment:
$\frac{27.22 - 20.54}{27.22} \approx 24.5\%$

42. $m = \int_0^{28}\left(\frac{1}{46} + \frac{x}{690}\right)^2 dx$
$+ \int_{28}^{30}\left(\frac{1}{92} + \frac{x}{690}\right)^2 dx$
≈ 0.05918 slugs.
$M = \int_0^{28} x\left(\frac{1}{46} + \frac{x}{690}\right)^2 dx$
$+ \int_{28}^{30} x\left(\frac{1}{92} + \frac{x}{690}\right)^2 dx$
≈ 1.1398 slugs
$\bar{x} = \frac{M}{m} \approx 19.258$
The center of mass moves in.

43. $\int_{-a}^{a} 2\rho x^2 b\sqrt{1 - \frac{x^2}{a^2}}dx = \frac{1}{4}\rho\pi a^3 b$

44. If the racket was solid wood, then the second moment would be
$M_0 = \int_{-a}^{a} 2\rho b x^2\sqrt{1 - \frac{x^2}{a^2}}\,dx = \rho\frac{\pi}{4}a^3 b$
But, the racket is not solid wood. We have to subtract the contribution to the second moment from the empty space. This amount is equal to the second moment of a smaller wood racket:
$M_1 = \int_{-(a-w)}^{a-w} 2\rho(b-w)x^2$
$\cdot\sqrt{1 - \frac{x^2}{(a-w)^2}}\,dx$
$= \rho\frac{\pi}{4}(a-w)^3(b-w)$
Therefore the second moment is
$M = M_0 - M_1$
$= \rho\frac{\pi}{4}\left[a^3 b - (a-w)^3(b-w)\right]$

45. Using the formula in Exercise 42, we find that the moments are 1323.8 for the wooden racket, 1792.9 for the mid-sized racket, and 2361.0 for the oversized racket. The ratios are
$\frac{\text{mid}}{\text{wood}} \approx 1.35$, $\frac{\text{over}}{\text{wood}} \approx 1.78$

46. $\frac{dM}{da} = \rho\frac{\pi}{4}\left[3a^2 b - 3(a-w)^2(b-w)\right]$
Since $a > a - w$ and $b > b - w$
$\frac{dM}{da} > 0.$
Therefore as a increases, M increases.
$\frac{dM}{dw} = \rho\frac{\pi}{4}\left[3(a-w)^2(b-w) + (a-w)^3\right]$
It is easy to see that $\frac{dM}{dw} > 0$. Therefore as w increases M increases making the racket more stable.

5.7 Probability

1. $f(x) = 4x^3 \geq 0$ for $0 \leq x \leq 1$ and
$\int_0^1 4x^3 dx = x^4\Big|_0^1 = 1 - 0 = 1$

2. $f(x) = \frac{3}{8}x^2 \geq 0$ on the interval $[0, 2]$ and
$\int_0^2 \frac{3}{8}x^2 dx = 1.$

3. $f(x) = x + 2x^3 \geq 0$ for $0 \leq x \leq 1$ and
$\int_0^1 (x + 2x^3)dx = \frac{x^2}{2} + \frac{x^4}{2}\Big|_0^1 = 1$

4. $f(x) = \cos x \geq 0$ over $[0, \pi/2]$ and
$\int_0^{\pi/2} \cos x dx = 1.$

5. $f(x) = \frac{1}{2}\sin x \geq 0$ over $[0, \pi]$ and
$\int_0^{\pi} \frac{1}{2}\sin x dx = \frac{1}{2} - \cos x\Big|_0^{\pi} = 1.$

6. $f(x) = e^{-x/2} \geq 0$ over $[0, \ln 4]$ and
$\int_0^{\ln 4} e^{-x/2}dx = -2e^{-x/2}\Big|_0^{\ln 4} = 1.$

7. We solve for c:
$1 = \int_0^1 cx^3 dx = \frac{c}{4}$ which gives $c = 4$.

8. We solve for c:
$1 = \int_0^1 cx + x^2 dx = \frac{c}{2} + \frac{1}{3}$
which gives $c = \frac{4}{3}$.

9. We solve for c:
$1 = \int_0^1 ce^{-4x} dx = -\frac{c}{4}(e^{-4} - 1)$
which gives $c = \frac{4}{1 - e^{-4}}$.

10. We solve for c:
$1 = \int_0^2 2ce^{-cx} dx = 2 - 2e^{-2c}$
which gives $c = \frac{1}{2}\ln 2$.

11. We solve for c:
$1 = \int_0^1 \frac{c}{1+x^2} = c\,\tan^{-1} x\Big|_0^1$
$= c\left(\frac{\pi}{4} - 0\right) = c\frac{\pi}{4}$
which gives $c = \frac{4}{\pi} \approx 1.2732$

12. We solve for c:
$1 = \int_0^1 \frac{c}{\sqrt{1-x^2}} = c\,\sin^{-1} x\Big|_0^1$
$= c\left(\frac{\pi}{2} - 0\right) = c\frac{\pi}{2}$
$\Rightarrow \; c = \frac{2}{\pi} \approx 0.6366$

13. $P(70 \le x \le 72)$
$= \int_{70}^{72} \frac{.4}{\sqrt{2\pi}} e^{-.08(x-68)^2} dx \approx 0.157$

14. $P(76 \le X \le 80)$
$= \int_{76}^{80} \frac{0.4}{\sqrt{2\pi}} e^{-0.08(x-68)^2} dx \approx 0.00068634$

15. $P(84 \le x \le 120)$
$= \int_{84}^{120} \frac{.4}{\sqrt{2\pi}} e^{-.08(x-68)^2} dx \approx 7.76 \times 10^{-11}$

16. $P(14 \le X \le 60)$
$= \int_{14}^{60} \frac{0.4}{\sqrt{2\pi}} e^{-0.08(x-68)^2} dx \approx 0.00068714$

17. $P\left(0 \le x \le \frac{1}{4}\right) = \int_0^{1/4} 6e^{-6x} dx$
$= -e^{-6x}\Big|_0^{1/4} = (-e^{-3/2} + 1) \approx .77687$

18. $P(0 \le X \le 0.5) = \int_0^{0.5} 6e^{-6x} dx \approx 0.95021$

19. $P(1 \le x \le 2) = \int_1^2 6e^{-6x} dx$
$= -e^{-6x}\Big|_1^2 = (-e^{-12} + e^{-6}) \approx .00247$

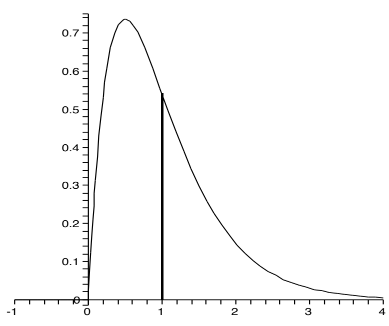
20. $P(3 \le X \le 10) = \int_3^{10} 6e^{-6x} dx$
$\approx 1.52300 \times 10^{-8}$

21. $P(0 \le x \le 1) = \int_0^1 4xe^{-2x} dx$
$= 1 - 3e^{-2} \approx .594$

22. $P(1 \le X \le 2) = \int_1^2 4xe^{-2x} dx \approx 0.31443$

23. Mean: $\int_0^{10} x(4xe^{-2x}) dx \approx 0.9999995$

24. The maximum is at $x = \frac{1}{2}$ and the mean is at $x \approx 0.31443$.



25. (a) Mean: $\mu = \int_a^b xf(x)dx = \int_0^1 3x^3 dx$
$= \frac{3}{4} = 0.75$

(b) Median, we must solve for m:
$\frac{1}{2} = \int_a^m f(x)dx = \int_0^m 3x^2 dx = m^3$
which gives $m = \frac{1}{\sqrt[3]{2}} \approx 0.7937$.

26. (a) Mean: $\mu = \int_a^b xf(x)dx = \int_0^1 4x^4 dx$
$= \frac{4}{5} = 0.8$

(b) Median, we must solve for m:
$\frac{1}{2} = \int_a^m f(x)dx = \int_0^m 4x^3 dx = m^4$
which gives $m = \frac{1}{\sqrt[4]{2}} \approx 0.8409$.

27. (a) Mean: $\mu = \int_a^b xf(x)dx$

$= \int_0^1 x\left(\frac{4/\pi}{1+x^2}\right)dx \approx 0.4413$

(b) Median, we must solve for m:

$\frac{1}{2} = \int_a^m f(x)dx$

$= \int_0^m \left(\frac{4/\pi}{1+x^2}\right)dx$

$= \frac{4}{\pi}\left(\tan^{-1}x\right)\Big|_0^m$

$= \frac{4}{\pi}\tan^{-1}m$

$\Rightarrow m = \tan\frac{\pi}{8} \approx 0.4142$

28. (a) Mean: $\mu = \int_a^b xf(x)dx$

$= \int_0^1 x\left(\frac{2/\pi}{\sqrt{1-x^2}}\right)dx$

≈ 0.6366

(b) Median, we must solve for m:

$\frac{1}{2} = \int_a^m f(x)dx$

$= \int_0^m \left(\frac{2/\pi}{\sqrt{1-x^2}}\right)dx$

$= \frac{2}{\pi}\sin^{-1}x\Big|_0^m$

$= \frac{2}{\pi}\left(\sin^{-1}m - 0\right)$

$= \frac{2}{\pi}\sin^{-1}m$

$\Rightarrow m = \sin\frac{\pi}{4} \approx 0.7071$

29. (a) Mean: $\mu = \int_a^b xf(x)dx$

$= \int_0^\pi \frac{1}{2}x\sin x dx$

$= \frac{1}{2}(\sin x - x\cos x)\Big|_0^\pi = \frac{\pi}{2}$

(b) Median, we must solve for m:

$\frac{1}{2} = \int_a^m f(x)dx$

$= \int_0^m \sin x dx = \frac{1}{2}(1-\cos m)$

which gives

$m = \cos^{-1}(0) = \frac{\pi}{2} \approx 1.57.$

30. (a) Mean: $\mu = \int_a^b xf(x)dx$

$= \int_0^{\pi/2} x\cos x dx$

$= \frac{\pi}{2} - 1 \approx 0.57080$

(b) Median, we must solve for m:

$\frac{1}{2} = \int_a^m f(x)dx$

$= \int_0^m \cos x dx = \sin m$

which gives $m = \frac{\pi}{6} \approx 0.5236.$

31. Density $f(x) = ce^{-4x}, [0,b], b > 0$

$1 = \int_0^b ce^{-4x}dx$

$= -\frac{c}{4}e^{-4x}\Big|_0^b = -\frac{c}{4}\left(e^{-4b} - 1\right)$

$c = \frac{4}{1-e^{-4b}}$

As $b \to \infty$, $c \to 4$

32. From Exercise 31, $c = \frac{4}{1-e^{-4b}}$

$\mu = \int_0^b cxe^{-4x}dx$

$= \frac{c}{16}\left[1 - e^{-4b}(1+4b)\right]$

$= \frac{1-e^{-4b}(1+4b)}{4(1-e^{-4b})}$

Now, taking the limit,

$\lim_{b\to\infty} \mu = \frac{1}{4}$

33. Density $f(x) = ce^{-6x}, [0,b], b > 0$

$1 = \int_0^b ce^{-6x}dx$

$= \frac{-c}{6}e^{-6x}\Big|_0^b = -\frac{c}{6}\left(e^{-6b} - 1\right)$

$c = \frac{6}{1-e^{-6b}}$

As $b \to \infty$, $c \to 6$

$\mu = \int_0^b xce^{-6x}dx$

$= \frac{ce^{-6c}}{36}(-6x-1)\Big|_0^b$

$= \frac{ce^{-6b}}{36}(-6b-1) + \frac{c}{36}$

As $b \to \infty$, $\mu \to \frac{1}{6}$

34. $c = \frac{A}{1-e^{-ab}}$

$\mu = \frac{1-e^{-ab}(1+ab)}{a(1-e^{-ab})}$

$\lim_{b\to\infty} \mu = \frac{1}{a}$

35. To find the probability of these events, we add the probabilities.

(a) $P(X \geq 5) = 0.0514 + 0.0115 + 0.0016 + 0.0001 = 0.0646$

(b) $P(X \leq 4) = 0.0458 + 0.1796 + 0.2953 + 0.2674 + 0.1473$
$= 0.9354$

(c) $P(X \geq 6) = 0.0115 + 0.0016 + 0.0001$
$= 0.0132$

(d) $P(X = 3 \text{ or } X = 4)$
$= 0.2674 + 0.1473$
$= 0.4147$

36. (a) $P(X = 2 \text{ or } X = 3) = 0.441 + 0.343$
$= 0.784$

(b) $P(X \geq 1) = 0.189 + 0.441 + 0.343 = 0.973$

37. (a) Suppose the statement is not true. Then there must be a game before which the player's winning percentage is smaller than 75% and after which the player's winning percentage is greater than 75%. Then there are integers a and b (note that $a \geq m, b \geq n$ and $a - b = m - n$), such that
$\frac{a}{b} < \frac{3}{4}$ and $\frac{a+1}{b+1} > \frac{3}{4}$. Then
$4a < 3b$, and $4a + 4 > 3b + 3$
$3b + 4 > 4a + 4 > 3b + 3$.

But there is no integer between the two numbers $3b+4$ and $3b+3$, and thus such situation will never happen. Thus there must be a game after which the player's winning percentage is exactly 75%.

(b) Using the same argument as in the previous problem, we can conclude that:

If after a certain game, a game player's winning percentage is strictly less than $100\frac{k}{k+1}$, and then the player wins several games in a row so that the winning percentage exceeds $100\frac{k}{k+1}$, then at some point in this process the player's winning percentage is exactly $100\frac{k}{k+1}$.

38. First the first quartile, we solve
$0.25 = \int_0^c \ln 2 e^{-(\ln 2)x/2} dx$
$= 2\left(1 - e^{-(\ln 2)c/2}\right)$
Solving gives
$c = -2\ln(7/8)/\ln 2 \approx 0.3853$ days.

For the third quartile, we solve
$0.75 = \int_0^c \ln 2 e^{-(\ln 2)x/2} dx$
$= 2\left(1 - e^{-(\ln 2)c/2}\right)$
Solving gives
$c = -2\ln(5/8)/\ln 2 \approx 1.3561$ days.

39. $f(x) = \frac{.4}{\sqrt{2\pi}} e^{-.08(x-68)^2}$

$f'(x) = \frac{-.064}{\sqrt{2\pi}}(x-68)e^{-.08(x-68)^2}$

$f''(x) = \frac{-.064}{\sqrt{2\pi}} e^{-.08(x-68)^2}$

$\cdot \left(1 - .16(x-68)^2\right)$

The second derivative is zero when
$x - 68 = \pm 1/\sqrt{0.16} = \pm 1/0.4 = \pm 5/2$
Thus the standard deviation is $\frac{5}{2}$.

40. For this, we have $\mu = 68$ and $\sigma = \frac{5}{2}$.
$P(\mu - \sigma \leq X \leq \mu + \sigma)$
$= P(65.5 < X < 70.5) \approx 0.6827$
$P(\mu - 2\sigma \leq X \leq \mu + 2\sigma)$
$= P(63 < X < 73) \approx 0.9545$
$P(\mu - 3\sigma \leq X \leq \mu + 3\sigma)$
$= P(60.5 < X < 75.5) \approx 0.9973$

41. $f'(p) = mp^{m-1}(1-p)^{n-m}$
$\quad - (n-m)p^m(1-p)^{n-m-1}$

$f'(p) = 0$ when $p = \frac{m}{n}$ and

$$f'(p) \begin{cases} < 0 & \text{if } p < m/n \\ > 0 & \text{if } p > m/n \end{cases}$$

Hence $f(p)$ is maximized when $p = \frac{m}{n}$.
In common senses, in order for an event to happen m times in n tries, the probability of the event itself should be about m/n.

42. In the picture, although it might appear that $y > 1/2$, the conditions are that $0 \leq y \leq 1/2$, and the labeling in the drawing implies that the lower line is the closer. This is indeed always an allowable assumption (by turning the picture upside down if necessary). In the right triangle whose hypotenuse is the lower half-needle, the vertical side is of length $(\sin\theta)/2$. Therefore the needle hits the lower line if $y - (\sin\theta)/2 \leq 0$, or if $y \leq (\sin\theta)/2$. As to the actual probability ratio, the denominator is just $\pi/2$, while the numerator is

$$-\frac{\cos\theta}{2}\bigg|_0^\pi = \frac{-\cos\pi + \cos 0}{2} = \frac{2}{2} = 1.$$

The total probability of hitting a line is thus $2/\pi \approx 63.66\%$.

43. To find the maximum, we take the derivative and set it equal to zero:
$f'(x) = -2ax(bx-1)(bx+1)e^{-b^2x^2} = 0$. This gives critical numbers $x = 0, \pm\frac{1}{b}$.
Since this will be a pdf for the interval $[0, 4m]$, we only have to check that there is a maximum at $\frac{1}{b}$. An easy check shows that
$f'(x) > 0$ on the interval $\left[0, \frac{1}{b}\right]$ and
$f'(x) < 0$ for $x > \frac{1}{b}$. Therefore there is a maximum at $x = m = \frac{1}{b}$ (the most common speed).

To find a in terms of m, we want the total probability equal to 1. Since $m = \frac{1}{b}$, we also make the substitution $b = \frac{1}{m}$.
$1 = \int_0^{4m} ax^2 e^{-x^2/m^2}\,dx$
Solving for a gives
$a = \left(\int_0^{4m} x^2 e^{-x^2/m^2}\,dx\right)^{-1}$
Note: this integral is not expressible in terms of elementary functions, so we will leave it like this. Using a CAS, one can find that
$a \approx 2.2568m^{-3}$

44. $f(t) = t^{-3/2}e^{0.38t-100/t}$
$\int_0^{40} k \cdot f(t)dt = 1$ for $k = 0.000318$.
$\int_{20}^{30} 0.000318 \cdot f(t)dt \approx 0.0134$

45. The probability of a $2k$-goal game ending in a $k-k$ tie is
$(2k) = \frac{(2k)\cdots(k+1)}{(k)\cdots(1)}p^k(1-p)^k$
$f(2k) < f(2k-2)$ for general k.
$\frac{f(2k)}{f(2k-2)} = 2\frac{2k-1}{k}p\,(1-p)$
Here $\frac{2k-1}{k} = 2 - \frac{1}{k} < 2$.
On the other hand,
$\left(p - \frac{1}{2}\right)^2 \geq 0, p^2 - p + \frac{1}{4} \geq 0$
$p - p^2 \leq \frac{1}{4}, p(1-p) \leq \frac{1}{4}$
Now we get $\frac{f(2k)}{f(2k-2)} = 2\frac{2k-1}{k}p\,(1-p)$
$< 2 \cdot 2 \cdot \frac{1}{4} = 1$. So $f(2k) < f(2k-2)$. In other words, the probability of a tie is decreasing as the number of goals increases.

46. The probability HTT appears first is the mean of that probability over the four possibilities for the first two coin tosses.

Let P(HT) be the probability HTT appears first following HT.

Suppose the first two throws are HH. Then the third throw can be either H or T. If it's H, then we are back in the same position: the preceding two throws are HH. But if it's T, then player B has won. So the probability of player A winning in this case is 0. Putting the two possibilities for the third throw together, as a mean, the probability that player A wins following HH is:
$\mathrm{P}\,(\mathrm{HH}) = \frac{1}{2} \times \mathrm{P}\,(\mathrm{HH}) + \frac{1}{2} \times 0 = \frac{1}{2}\mathrm{P}\,(\mathrm{HH})$.
Now suppose the first two throws are HT. If the third throw is H, then neither player has won, and the probability HTT will ultimately win is (by definition) P(TH). (The last two throws were TH.) On the other hand, if the third throw is T, then player A has won! So this time the weighted mean for the probability that player A wins, following HT is:
$\mathrm{P}\,(\mathrm{HH}) = \frac{1}{2} \times \mathrm{P}\,(\mathrm{TH}) + \frac{1}{2} \times 1 = \frac{1}{2}\mathrm{P}\,(\mathrm{TH}) + \frac{1}{2}$
Similarly, we get
$\mathrm{P}\,(\mathrm{TH}) = \frac{1}{2} \times \mathrm{P}\,(\mathrm{HH}) + \frac{1}{2} \times \mathrm{P(HT)}$ and
$\mathrm{P}\,(\mathrm{TT}) = \frac{1}{2} \times \mathrm{P}\,(\mathrm{TH}) + \frac{1}{2} \times \mathrm{P(TT)}$.
Therefore, we have
P(HH) = 0
P(HT) = P(HT)/4 + 1/2 P(HT) = 2/3
P(TH) = P(HT)/2 = 1/3
P(TT) = P(TH) P(TT) = 1/3
The mean of these four results gives us the probability of HTT appearing before HHT is 1/3. Hence, the probability of HHT appearing before HTT is 2/3. Therefore, player B is twice as likely to win.

47. (a) The functions $f\,(x)$ and $g\,(x)$ are the pdfs, such that $f\,(x) = a + bx + cx^2$;
$f\,(x^2) = g\,(x)$.
Therefore by definition,
$f\,(x)\,; g\,(x) \geq 0$ and
$\int_0^1 f(x)dx = \int_0^1 g(x)dx = 1$
Consider $f(x) = a + bx + cx^2$ and
$g(x) = f(x^2) = a + bx^2 + cx^4$.

Thus, $1 = \int_0^1 f(x)dx$
$= \int_0^1 \left(a + bx + cx^2\right) dx$
$= \left(ax + b\frac{x^2}{2} + c\frac{x^3}{3}\right)\Big|_0^1$
$\Rightarrow a + \frac{b}{2} + \frac{c}{3} = 1.....(1)$
and $1 = \int_0^1 g(x)dx$
$= \int_0^1 \left(a + bx^2 + cx^4\right) dx$
$= \left(ax + b\frac{x^3}{3} + c\frac{x^5}{5}\right)\Big|_0^1$
$\Rightarrow a + \frac{b}{3} + \frac{c}{5} = 1.....(2)$
Solving (1) and (2), we get,
$b = -\frac{4c}{5}; a = 1 + \frac{c}{15};$
Thus $f(x) = 1 + \frac{c}{15} - \frac{4c}{5}x + cx^2$
or $f(x) = \frac{\left(15cx^2 - 12cx + c + 15\right)}{15}$

(b) Mean of pdf g:
$\mu = \int_a^b xg(x)dx$
$= \int_0^1 x\frac{\left(15cx^4 - 12cx^2 + c + 15\right)}{15}dx$
$= \frac{1}{15}\int_0^1 \left(15cx^5 - 12cx^3 + (c+15)\,x\right) dx$
$= \frac{1}{15}\left(\frac{15cx^6}{6} - \frac{12cx^4}{4} + \frac{(c+15)\,x^2}{2}\right)\Big|_0^1$
$= 0.5$

Ch. 5 Review Exercises

1. Area $= \int_0^\pi \left(x^2 + 2 - \sin x\right) dx$
$= \left(\frac{x^3}{3} + 2x + \cos x\right)\Big|_0^\pi$
$= \frac{\pi^3}{3} + 2\pi - 2$

2. Area $= \int_0^1 \left(e^x - e^{-x}\right) dx$
$= \left(e^x + e^{-x}\right)\Big|_0^1 = e + e^{-1} - 2$

3. Area $= \int_0^1 x^3 - \left(2x^2 - x\right)\, dx$
$= \left(\frac{x^4}{4} - \frac{2}{3}x^3 + \frac{x^2}{2}\right)\Big|_0^1 = \frac{1}{12}$

4. First solve $x^2 - 3 = -x^2 + 5$ to find that the intersections points are $x = -2, 2$.
Area $= \int_{-2}^2 [(-x^2 + 5) - (x^2 - 3)]\, dx$
$= \left(-\frac{2}{3}x^3 + 8x\right)\Big|_{-2}^2 = \frac{64}{3}.$

5. Solving $e^{-x} = 2 - x^2$ we get
$x \approx -0.537, 1.316$
Area $\approx \int_{-.537}^{1.316} \left(2 - x^2 - e^x\right) dx$
$= \left(2x - \frac{x^3}{3} + e^{-x}\right)\Big|_{-.537}^{1.316} \approx 1.452$

6. First solve $y^2 = 1 - y$ to find that the intersections points are $y = \frac{-1 \pm \sqrt{5}}{2}$.
Area $= \int_{\frac{-1-\sqrt{5}}{2}}^{\frac{-1+\sqrt{5}}{2}} [(1 - y) - y^2]\, dy$
$= \left(y - \frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_{\frac{-1-\sqrt{5}}{2}}^{\frac{-1+\sqrt{5}}{2}}$
$= \frac{5\sqrt{5}}{6}.$

7. Area $= \int_0^1 x^2\, dx + \int_1^2 (2 - x)\, dx$
$= \frac{x^3}{3}\Big|_0^1 + \left(2x - \frac{x^2}{2}\right)\Big|_1^2$
$= \frac{1}{3} + (4 - 2) - \left(2 - \frac{1}{2}\right) = \frac{5}{6}$

8. Area $= \int_0^2 x^2\, dx = \frac{8}{3}$

9. If P is the population at time t, the equation is
$P'(t) =$ birth rate $-$ death rate
$= (10 + 2t) - (4 + t) = 6 + t$
Thus $P = 6t + t^2/2 + P(0)$, so at time $t = 6$,

$$P(6) = 36 + 18 + 10{,}000 = 10{,}054.$$

Alternatively,

$A = \int_0^6 [(10 + 2t) - (4 + t)]dt$

$$= \int_0^6 (6+t)dt = \left(6t + \frac{t^2}{2}\right)\Bigg|_0^6 = 54$$

population $= 10{,}000 + 54 = 10{,}054$

10. For this we use Simpson's rule on the function $(f-g)(x)$.

$$\int_0^2 [f(x) - g(x)]\,dx$$

$\approx \frac{0.2}{3}[(3.2-1.2)+4(3.5-1.5)+2(3.8-1.6) + 4(3.7-2.2)+2(3.2-2.0)+4(3.4-2.4) +2(3.0-2.2)+4(2.8-2.1)+2(2.3-2.3) +4(2.9-2.8)+(3.4-2.4)]$
$\approx 2.1733.$

11. $V = \int_0^2 \pi(3+x)^2\,dx$

$$= \pi \int_0^2 (9+6x+x^2)\,dx$$
$$= \pi \left(9x + 3x^2 + \frac{x^3}{3}\right)\Bigg|_0^2$$
$$= \frac{98\pi}{3}$$

12. If we consider slices perpendicular to the x-axis, then the area of a slice is equal to $(10+2x)(4+x)$ (length times depth). We integrate the areas from $x=0$ to $x=2$:

Area $= \int_0^2 (10+2x)(4+x)\,dx$

$= \frac{364}{3} \approx 121.33$ cubic feet.

13. Use trapezoidal estimate:

$$V = 0.4\left(\frac{0.4}{2} + 1.4 + 1.8 + 2.0 + 2.1 + 1.8 + 1.1 + \frac{0.4}{2}\right)$$

≈ 4.2

14. (a) $V = \int_0^1 \pi x^4\,dx = \frac{\pi}{5}$

(b) $V = \int_0^1 \pi(1-y)\,dy = \frac{\pi}{2}$

(c) $V = \int_0^1 \pi[(2-\sqrt{y})^2 - 1]\,dy = \frac{5\pi}{6}$

(d) $V = \int_0^1 \pi[(2+x^2)^2 - 2]\,dx = \frac{53}{15}$

15. (a) $V = \int_{-2}^2 \pi(4)^2\,dx - \int_{-2}^2 \pi(x^2)^2\,dx$

$$= \pi \int_{-2}^2 (16 - x^4)\,dx$$
$$= \pi\left(16x - \frac{x^5}{5}\right)\Bigg|_{-2}^2$$
$$= \frac{256\pi}{5}$$

(b) $V = \int_0^4 \pi(\sqrt{y})^2 dy = \pi \int_0^4 y dy$

$$= \frac{\pi y^2}{2}\Bigg|_0^4 = 8\pi$$

(c) $V = \int_0^4 \pi(2+\sqrt{y})^2 dy$

$$- \int_0^4 \pi(2-\sqrt{y})^2 dy$$
$$= \pi \int_0^4 (4 + 4y^{1/2} + y)dy$$
$$- \pi \int_0^4 (4 - 4y^{1/2} + y)dy$$
$$= \pi \int_0^4 (8y^{1/2})dy$$
$$= 8\pi \cdot \frac{2}{3} y^{3/2}\Bigg|_0^4 = \frac{128\pi}{3}$$

(d) $V = \int_{-2}^2 \pi(6)^2\,dx$

$$- \int_{-2}^2 \pi(x^2+2)^2\,dx$$
$$= \pi \int_{-2}^2 (-x^4 - 4x^2 + 32)\,dx$$
$$= \pi\left(-\frac{x^5}{5} - \frac{4x^3}{3} + 32x\right)\Bigg|_{-2}^2$$
$$= \frac{1408\pi}{15}$$

16. (a) $V = \int_0^2 \pi(4x^2 - x^2)\,dx = 8\pi$

(b) $V = \int_0^2 \pi\left(y^2 - \frac{y^2}{4}\right) dy$

$$+ \int_2^4 \pi\left(4 - \frac{y^2}{4}\right) dy$$
$$= 2\pi + \frac{10\pi}{3} = \frac{16\pi}{3}$$

(c) V

$$= \int_0^2 \pi\left[(1+y)^2 - \left(1+\frac{y}{2}\right)^2\right] dy$$
$$+ \int_2^4 \pi\left[9 - \left(1+\frac{y}{2}\right)^2\right] dy$$
$$= 4\pi + \frac{16\pi}{3} = \frac{28\pi}{3}$$

(d) $V = \int_0^2 \pi[(4-x)^2 - (4-2x)^2]\,dx$
$= 8\pi$

17. (a) $V = \int_0^1 2\pi y((2-y)-y)dy$
$= 2\pi \int_0^1 (2y - 2y^2)dy$
$= 2\pi \left(y^2 - \frac{2y^3}{3}\right)\Big|_0^1$
$= \frac{2\pi}{3}$

(b) $V = \int_0^1 \pi(2-y)^2 dy$
$- \int_0^1 \pi(y)^2 dy$
$= \pi \int_0^1 (4-4y)dy$
$= \pi\ (4y - 2y^2)\big|_0^1 = 2\pi$

(c) $V = \int_0^1 \pi((2-y)+1)^2 dy$
$- \int_0^1 \pi(y+1)^2 dy$
$= \pi \int_0^1 (9 - 6y + y^2)dy$
$- \pi \int_0^1 (y^2 + 2y + 1)dy$
$= \pi \int_0^1 (8 - 8y)dy$
$= \pi\ (8y - 4y^2)\big|_0^1 = 4\pi$

(d) $V = \int_0^1 2\pi(4-y)((2-y)-y)dy$
$= 2\pi \int_0^1 (8 - 10y + 2y^2)dy$
$= 2\pi \left(8y - 5y^2 + \frac{2y^3}{3}\right)\Big|_0^1$
$= \frac{22\pi}{3}$

18. (a) Method of shells.
$V = \int_0^2 2\pi y[(4-y^2) - (y^2-4)]\,dy$
$= 16\pi$

(b) $V = \int_{-2}^2 \pi(4-y^2)^2\,dy = \frac{512\pi}{15}$

(c) $V = \int_{-2}^2 \pi[(8-y^2)^2 - y^4]\,dy$
$= \frac{512\pi}{3}$

(d) Method of shells.
$V = \int_{-2}^2 2\pi(4-y)[(4-y^2)$
$- (y^2-4)]\,dy$
$= \frac{208\pi}{3}$

19. $s = \int_{-1}^1 \sqrt{1 + (4x^3)^2}\,dx \approx 3.2$

20. $s = \int_{-1}^0 \sqrt{1 + (2x+1)^2}\,dx \approx 1.14779$

21. $s \int_{-2}^2 \sqrt{1 + \left(\frac{e^{x/2}}{2}\right)^2}\,dx \approx 4.767$

22. $s = \int_0^\pi \sqrt{1 + 4\cos^2 2x}\,dx \approx 5.27037$

23. $S = \int_0^1 2\pi(1-x^2)\sqrt{1+4x^2}\,dx$
≈ 5.483

24. $S = \int_0^1 2\pi x^3 \sqrt{1+9x^4}\,dx \approx 3.56312$

25. $h''(t) = -32$
$h(0) = 64, h'(0) = 0$
$h'(t) = -32t$
$h(t) = -16t^2 + 64$

This is zero when $t = 2$, at which time $h'(2) = -32(2) = -64$. The speed at impact is reported as 64 feet per second.

26. In this case we have the equations
$h''(t) = -32$
$h(0) = 64 \quad h'(0) = 4$
$h'(t) = -32t + 4$
$h(t) = -16t^2 + 4t + 64$

This is zero when
$t = t_0 = \frac{1 + \sqrt{257}}{8}$

Therefore the velocity at impact is
$h'(t_0) = \frac{-32(1+\sqrt{257})}{8} + 4$
$= -4\sqrt{257} \approx -64.125$ ft/s

27. $y''(t) = -32, x''(t) = 0,$
$y(0) = 0, x(0) = 0$
$y'(0) = 48 \sin\left(\frac{\pi}{9}\right)$
$x'(0) = 48 \cos\left(\frac{\pi}{9}\right)$
$y'(0) \approx 16.42, x'(0) \approx 45.11$

$y'(t) = -32t + 16.42$
$y(t) = -16t^2 + 16.42t$

This is zero at $t = 1.026$. Meanwhile,

$x'(t) \equiv 45.11$
$x(t) = 45.11t$
$x(1.026) = 45.11(1.026) \approx 46.3$ ft This is the horizontal range.

28. In this case we have the equations

$y''(t) = -32, x''(t) = 0$
$y(0) = 6, x(0) = 0$
$y'(0) = 48\sin\frac{\pi}{9}, \quad x'(0) = 48\cos\frac{\pi}{9}$
$y'(t) = -32t + 48\sin\frac{\pi}{9}$
$x'(t) = 48\cos\frac{\pi}{9}$
$y(t) = -16t^2 + 48t\sin\frac{\pi}{9} + 6$
$x(t) = 48t\cos\frac{\pi}{9}$

We now solve $y(t) = 0$ or

$-16t^2 + 48t\sin\frac{\pi}{9} + 6 = 0$

which gives $t \approx 1.3119$, this is the time of flight.

The horizontal range is

$x(1.3119) \approx 59.17$ feet.

29. $y(0) = 6, x(0) = 0$
$y'(0) = 80\sin\left(\frac{2\pi}{45}\right) \approx 11.13,$
$x'(0) = 80\cos\left(\frac{2\pi}{45}\right) \approx 79.22$
$y''(t) = -32, x''(t) = 0$
$y'(t) = -32t + 11.13$
$y(t) = -16t^2 + 11.13t + 6$
$x'(t) = 79.22$
$x(t) = 79.22t$

This is 120 (40 yards) when t is about 1.51. At this time, the vertical height (if still in flight) would be
$y(1.51) = -16(1.51)^2 + 11.13(1.51) + 6$
$= -13.6753,$

Since this is negative, we conclude the ball is not still in flight, has hit the ground, and was not catchable.

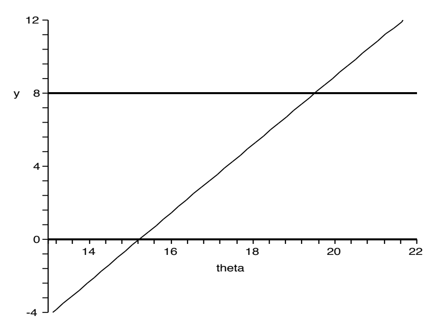
30. If we repeat Exercise 29, but we'll leave the angle as θ (we will plug in $\theta = 24° = \frac{2\pi}{15}$ later too).

Our equations become
$y(0) = 6, \quad x(0) = 0$
$y'(0) = 80\sin\theta, \quad x'(0) = 80\cos\theta$
$y''(t) = -32, \quad x''(t) = 0$

Integrating and using the initial conditions gives
$y'(t) = -32t + 80\sin\theta$
$x'(t) = 80\cos\theta$
$y(t) = -16t^2 + 80t\sin\theta + 6$
$x(t) = 80t\cos\theta$

We solve for the time when the ball is 40 yards down the field:
$120 = x(t) = 80t\cos\theta$
Solving gives
$t_0 = t = \frac{3}{2}\sec\theta$
The height at this time is
$y(t_0) = -16\left(\frac{3}{2}\sec\theta\right)^2$
$\quad + 80\left(\frac{3}{2}\sec\theta\right)\sin\theta + 6$
$= -36\sec^2\theta + 120\tan\theta + 6$

Let us say that the ball is catchable if it is between 0 and 8 feet high when the ball reaches the 40 yard point (the player can dive or jump to catch a low or high ball). To determine when this occurs, we graph the function and see that for the ball to be catchable it must be thrown with angle in the range:
$15.23° < \theta < 19.51°$



31. $h''(t) = -32$
$h'(0) = v_0$
$h(0) = 0$
$h'(t) = -32t + v_0$
This is zero at $t = v_0/32$.
$h\left(\frac{v_0}{32}\right) = -16\left(\frac{v_0^2}{32^2}\right) + \frac{v_0^2}{32} = \frac{v_0^2}{64}$
If this is to be 128, then clearly v_0 must be

$$\sqrt{(64)(128)} = 64\sqrt{2} \text{ ft/sec.}$$

Impact speed from ground to ground is the same as launch speed, which can be verified by first finding the time t of return to ground:
$-16t^2 + v_0t = 0$

$t = v_0/16$
and then compiling

$$h'(v_0/16) = -32(v_0/16) + v_0 = -v_0$$

32. We want to determine how far in the x-direction the drop travels. We have initial conditions
$x'(0) = 100$, $x(0) = 0$
$y'(0) = 0$, $y(0) = 120$
$x'(t) = 100$, $x(t) = 100t$,
$y'(t) = -32t$, $y(t) = -16t^2 + 120$

We first solve $0 = y = -16t^2 + 120$ to get $t = \sqrt{\frac{15}{2}}$. This is when the supplies hit the ground. We plug this into the equation $x(t)$ to determine how far the supplies traveled.

$$x\left(\sqrt{\frac{15}{2}}\right) = 100\sqrt{\frac{15}{2}} \approx 273.86$$

So, the supplies should be dropped 273.86 feet before the target.

33. $F = kx,\ 60 = k \cdot 1,\ k = 60$

$W = \int_0^{2/3} 60x\,dx = 30x^2\Big|_0^{2/3}$

$= \frac{30 \cdot 4}{9} = \frac{40}{3}$ ft-lb

34. Remember to convert miles to feet.

$W = \int_0^8 (800 + 2x)\,dx$

$= 6464$ mile-pounds
$= 3.413 \times 10^7$ foot-pounds.

35. $m = \int_0^4 \left(x^2 - 2x + 8\right) dx$

$= \left(\frac{x^3}{3} - x^2 + 8x\right)\Big|_0^4 = \frac{112}{3}$

$M = \int_0^4 x\left(x^2 - 2x + 8\right)\,dx$

$= \int_0^4 \left(x^3 - 2x^2 + 8x\right) dx$

$= \left(\frac{x^4}{4} - \frac{2x^3}{3} + 4x^2\right)\Big|_0^4 = \frac{256}{3}$

$\bar{x} = \frac{M}{m} = \frac{\frac{256}{3}}{\frac{112}{3}} = \frac{256}{112} = \frac{16}{7}$

Center of mass is greater than 2 because the object has greater density on the right side of the interval $[0, 4]$.

36. $m = \int_0^2 (x^2 - 2x + 8)\,dx = \frac{44}{3}$.

$M = \int_0^2 x(x^2 - 2x + 8)\,dx = \frac{44}{3}$.

$\overline{x} = \frac{M}{m} = 1$

The center of mass is at one because the density function is symmetrical about the point $x = 1$. (The graph of $y = x^2 - 2x + 8$ is a parabola with vertex at $x = 1$.)

37. $F = \int_0^{80} 62.4x(140 - x)\,dx$

$= 62.4 \int_0^{80} (140x - x^2)\,dx$

$= 62.4\left(70x^2 - \frac{x^3}{3}\right)\Big|_0^{80}$

$= 62.4(80)^2(130/3)$

$\approx 17{,}305{,}600$ lb

38. $F = \int_5^{10} 62.4(20)x\,dx = 46800$ lb

39. $J \approx \frac{.0008}{3(8)}\{0 + 4(800) + 2(1600)$

$+ 4(2400) + 2(3000) + 4(3600)$
$+2(2200) + 4(1200) + 0\}$
$= 1.52$

$J = m\Delta v$

$1.52 = .01\Delta v$
$\Delta v = 152$ ft/s
$152 - 120 = 32$ ft/s

40. $J = \int_0^2 3000t(2 - t)\,dt = 4000$

Since $J = m\Delta v$, we have $\Delta v = \frac{4000}{100} = 40$ and the speed before the collision must have been 40 feet per second (about 23.7 miles per hour).

41. $f(x) = x + 2x^3$ on $[0, 1]$
$f(x) \geq 0$ for $0 \leq x \leq 1$ and

$$\int_0^1 \left(x + 2x^3\right) dx = \left(\frac{x^2}{2} + \frac{x^4}{2}\right)\Big|_0^1 = 1$$

42. The function is positive on the interval, and

$$\int_0^{\ln 2} \frac{8}{3}e^{-2x}\,dx = 1.$$

43.
$$1 = \int_1^2 \frac{c}{x^2}\,dx = \left.\frac{-c}{x}\right|_1^2 = \frac{-c}{2} + c = \frac{c}{2}$$
Therefore $c = 2$

44. We want to solve for c:
$$1 = \int_0^4 ce^{-2x}\,dx = \frac{c}{2}(1 - e^{-8})$$
Solving gives
$$c = \frac{2}{1 - e^{-8}}.$$

45. (a) $P(x < .5) = \int_0^{.5} 4e^{-4x}\,dx$
$= -e^{-4x}\big|_0^{.5} = 1 - e^{-2} \approx .864$

(b) $P(.5 \le x \le 1) = \int_{.5}^1 4e^{-4x}\,dx$
$= -e^{-4x}\big|_{.5}^1 = -e^{-4} + e^{-2} \approx .117$

46. (a) $P\left(X < \frac{1}{12}\right) = \int_0^{1/12} 9xe^{-3x}\,dx$
$= 1 - \frac{5}{4}e^{-1/4} \approx 0.026499$

(b) $P\left(\frac{1}{2} < X < 1\right) = \int_{1/2}^1 9xe^{-3x}\,dx$
$= \frac{5}{2}e^{-3/2} - 4e^{-3} \approx 0.35868$

47. (a) $\mu = \int_0^1 x\left(x + 2x^3\right)\,dx$
$= \left.\frac{x^3}{3} + \frac{2x^5}{5}\right|_0^1 = \frac{11}{15} \approx 0.7333$

(b) $\frac{1}{2} = \int_0^c \left(x + 2x^3\right)dx$
$= \left.\frac{x^2}{2} + \frac{x^4}{2}\right|_0^c = \frac{c^2}{2} + \frac{c^4}{2}$
Therefore $c^2 + c^4 = 1$,
$$c = \sqrt{\frac{-1+\sqrt{5}}{2}} \approx 0.786$$

48. (a) $\mu = \int_0^{\ln 2} \frac{8}{3}xe^{-2x}\,dx$
$= \frac{1}{2} - \frac{1}{3}\ln 2 \approx 0.26895$

(b) For the median, we have to solve the equation
$$0.5 = \int_0^m \frac{8}{3}e^{-2x}\,dx = \frac{4}{3}(1 - e^{2m})$$
Solving gives
$$m = \frac{1}{2}\ln(8/5) \approx 0.23500$$