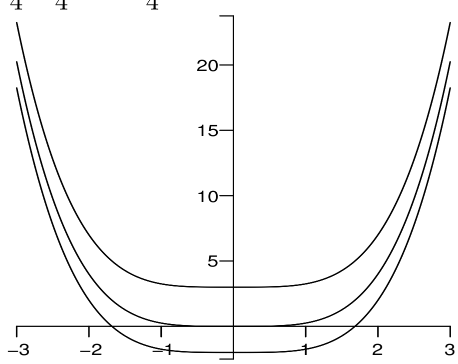
Chapter 4

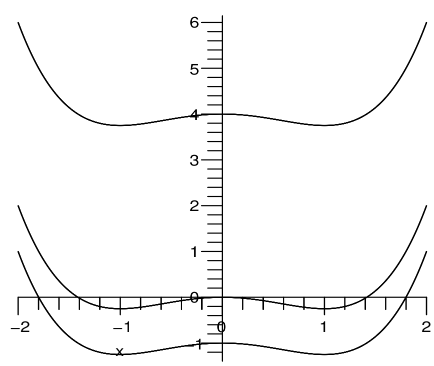
Integration

4.1 Antiderivatives

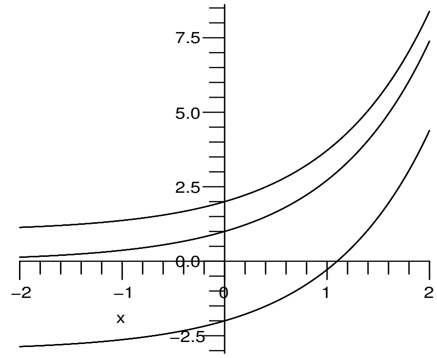
1. $\dfrac{x^4}{4}, \dfrac{x^4}{4}+3, \dfrac{x^4}{4}-2$



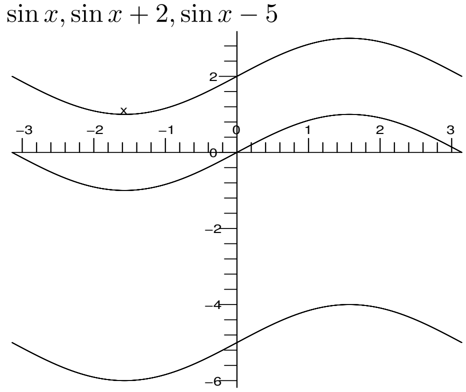
2. $\dfrac{x^4}{4}-\dfrac{x^2}{2}, \dfrac{x^4}{4}-\dfrac{x^2}{2}-1, \dfrac{x^4}{4}-\dfrac{x^2}{2}+4$



3. e^x, e^x+1, e^x-3



4. $\sin x, \sin x+2, \sin x-5$



5. $\displaystyle\int (3x^4-3x)dx = \frac{3}{5}x^5 - \frac{3}{2}x^2 + c$

6. $\displaystyle\int (x^3-2)dx = \frac{1}{4}x^4 - 2x + c$

7. $\displaystyle\int \left(3\sqrt{x} - \frac{1}{x^4}\right) dx = 2x^{3/2} + \frac{x^{-3}}{3} + c$

8. $\displaystyle\int \left(2x^{-2} + \frac{1}{\sqrt{x}}\right) dx$
$= -2x^{-1} + 2x^{1/2} + c$

9. $\displaystyle\int \frac{x^{1/3}-3}{x^{2/3}}dx = \int (x^{-1/3} - 3x^{-2/3})dx$
$= \dfrac{3}{2}x^{2/3} - 9x^{1/3} + c$

10. $\displaystyle\int \frac{x+2x^{3/4}}{x^{5/4}}dx = \int (x^{-1/4} + 2x^{-1/2})dx$
$= \dfrac{4}{3}x^{3/4} + 4x^{1/2} + c$

11. $\displaystyle\int (2\sin x + \cos x)dx = -2\cos x + \sin x + c$

12. $\displaystyle\int (3\cos x - \sin x)dx = 3\sin x + \cos x + c$

13. $\displaystyle\int 2\sec x \tan x dx = 2\sec x + c$

14. $\int \frac{4}{\sqrt{1-x^2}}dx = 4\arcsin x + c$

15. $\int 5\sec^2 x dx = 5\tan x + c$

16. $\int \frac{4\cos x}{\sin^2 x}dx = -4\csc x + c$

17. $\int (3e^x - 2)dx = 3e^x - 2x + c$

18. $\int (4x - 2e^x)dx = 2x^2 - 2e^x + c$

19. $\int (3\cos x - 1/x)dx = 3\sin x - \ln|x| + c$

20. $\int (2x^{-1} + \sin x)dx = 2\ln|x| - \cos x + c$

21. $\int \frac{4x}{x^2+4}dx = 2\ln|x^2+4| + c$

22. $\int \frac{3}{4x^2+4}dx = \frac{3}{4}\tan^{-1} x + c$

23. $\int \frac{\cos x}{\sin x}dx = \ln|\sin x| + c$

24. $\int (2\cos x - e^x)dx = 2\sin x - e^x + c$

25. $\int \frac{e^x}{e^x+3}dx = \ln|e^x + 3| + c$

26. $\int \frac{e^x+3}{e^x}dx = \int (1 + 3e^{-x})dx$
$= x - 3e^{-x} + c$

27. $\int x^{1/4}(x^{5/4} - 4)dx = \int (x^{3/2} - 4x^{1/4})dx$
$= \frac{2}{5}x^{5/2} - \frac{16}{5}x^{5/4} + c$

28. $\int x^{2/3}(x^{-4/3} - 3)dx = \int (x^{-2/3} - 3x^{2/3})dx$
$= 3x^{1/3} - \frac{9}{5}x^{5/3} + c$

29. $\frac{d}{dx}\ln|\sec x + \tan x|$
$= \frac{1}{\sec x + \tan x}\frac{d}{dx}(\sec x + \tan x)$
$= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$
$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$
$= \sec x$

30. $\frac{d}{dx}\ln|\sin x \cdot 2|$
$= \frac{1}{\sin x \cdot 2}\frac{d}{dx}(\sin x \cdot 2)$
$= \frac{2\cos x}{2\sin x} = \cot x$

31. (a) N/A
(b) By Power Formula,
$$\int (\sqrt{x^3} + 4)dx = \frac{2}{5}x^{5/2} + 4x + c.$$

32. (a) By Power Formula,
$\int \frac{3x^2-4}{x^2}dx = \int (3 - 4x^{-2})dx$
$= 3x + 4x^{-1} + c$
(b) N/A

33. (a) N/A
(b) By Reversing derivative formula,
$\int \sec^2 x dx = \tan x + c$

34. (a) By Power Formula,
$\int \left(\frac{1}{x^2} - 1\right)dx = -\frac{1}{x} - x + c$
(b) N/A

35. Finding the antiderivative,
$f(x) = 3e^x + \frac{x^2}{2} + c.$
Since $f(0) = 4$,
we have $4 = f(0) = 3 + c$.
Therefore,
$f(x) = 3e^x + \frac{x^2}{2} + 1.$

36. Finding the antiderivative,
$f(x) = 4\sin x + c.$
Since $f(0) = 3$,
we have $3 = f(0) = c$.
Therefore,
$f(x) = 4\sin x + 3.$

37. Finding the antiderivative
$f'(x) = 4x^3 + 2e^x + c_1.$
Since, $f'(0) = 2$.
We have $2 = f'(0) = 2 + c_1$
and therefore
$f'(x) = 4x^3 + 2e^x.$
Finding the antiderivative,
$f(x) = x^4 + 2e^x + c_2.$
Since $f(0) = 3$,
We have $3 = f(0) = 2 + c_2$
Therefore,

$f(x) = x^4 + 2e^x + 1.$

38. Finding the antiderivative,
$f'(x) = 5x^4 + e^{2x} + c_1.$
Since $f'(0) = -3$,
we have $-3 = f'(0) = 1 + c_1$
Therefore,
$f'(x) = 5x^4 + e^{2x} - 4.$
Finding the antiderivative,
$f(x) = x^5 + \dfrac{e^{2x}}{2} - 4x + c_2.$
Since $f(0) = 2$,
We have $2 = f(0) = \dfrac{1}{2} + c_2$
Therefore,
$f(x) = x^5 + \dfrac{e^{2x}}{2} - 4x + \dfrac{3}{2}.$

39. Taking antiderivatives,
$f'(t) = 2t + t^2 + c_1$
$f(t) = t^2 + \dfrac{t^3}{3} + c_1 t + c_2$
Since $f(0) = 2$,
we have $2 = f(0) = c_2$
Therefore,
$f(t) = t^2 + \dfrac{t^3}{3} + c_1 t + 2.$
Since $f(3) = 2$,
we have
$2 = f(3) = 9 + 9 + 3c_1 + 2$
$-6 = c_1$
Therefore,
$f(t) = \dfrac{t^3}{3} + t^2 - 6t + 2.$

40. Taking antiderivatives,
$f'(t) = 4t + 3t^2 + c_1$
$f(t) = 2t^2 + t^3 + c_1 t + c_2$
Since $f(1) = 3$,
we have $3 = f(1) = 2 + 1 + c_1 + c_2$
Therefore,
$c_1 + c_2 = 0$
Since $f(-1) = -2$,
we have $-2 = f(-1) = 2 - 1 - c_1 + c_2$
Therefore, $-c_1 + c_2 = -3$.
So, $c_1 = \frac{3}{2}$ and $c_2 = -\frac{3}{2}$
Hence,
$f(t) = t^3 + 2t^2 + \dfrac{3}{2}t - \dfrac{3}{2}.$

41. Taking antiderivatives,
$f''(x) = 3\sin x + 4x^2$
$f'(x) = -3\cos x + \dfrac{4}{3}x^3 + c_1$
$f(x) = -3\sin x + \dfrac{1}{3}x^4 + c_1 x + c_2.$

42. Taking antiderivatives,
$f''(x) = x^{1/2} - 2\cos x$
$f'(x) = \dfrac{2}{3}x^{3/2} - 2\sin x + c_1$
$f(x) = \dfrac{4}{15}x^{5/2} + 2\cos x + c_1 x + c_2.$

43. Taking antiderivatives,
$f'''(x) = 4 - 2/x^3$
$f''(x) = 4x + x^{-2} + c_1$
$f'(x) = 2x^2 - x^{-1} + c_1 x + c_2$
$f(x) = \dfrac{2}{3}x^3 - \ln|x| + \dfrac{c_1}{2}x^2 + c_2 x + c_3$

44. Taking antiderivatives,
$f'''(x) = \sin x - e^x$
$f''(x) = -\cos x - e^x + c_1$
$f'(x) = -\sin x - e^x + c_1 x + c_2$
$f(x) = \cos x - e^x + \dfrac{c_1}{2}x^2 + c_2 x + c_3$

45. Position is the antiderivative of velocity,
$s(t) = 3t - 6t^2 + c.$
Since $s(0) = 3$, we have $c = 3$. Thus,
$s(t) = 3t - 6t^2 + 3.$

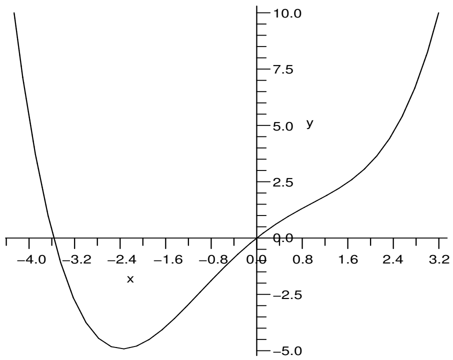
46. Position is the antiderivative of velocity,
$s(t) = -3e^{-t} - 2t + c.$
Since $s(0) = 0$, we have $-3 + c = 0$ and therefore $c = 3$. Thus,
$s(t) = -3e^{-t} - 2t + 3.$

47. First we find velocity, which is the antiderivative of acceleration,
$v(t) = -3\cos t + c_1.$
Since $v(0) = 0$ we have
$-3 + c_1 = 0$, $c_1 = 3$ and
$v(t) = -3\cos t + 3.$
Position is the antiderivative of velocity,
$s(t) = -3\sin t + 3t + c_2.$
Since $s(0) = 4$, we have $c_2 = 4$. Thus,
$s(t) = -3\sin t + 3t + 4.$

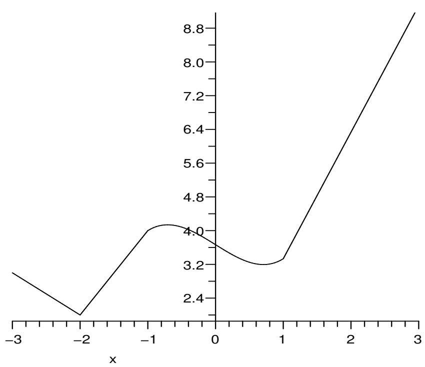
48. First we find velocity, which is the antiderivative of acceleration,
$v(t) = \dfrac{1}{3}t^3 + t + c_1.$
Since $v(0) = 4$ we have $c_1 = 4$ and
$v(t) = \dfrac{1}{3}t^3 + t + 4.$

Position is the antiderivative of velocity,
$s(t) = \dfrac{1}{12}t^4 + \dfrac{1}{2}t^2 + 4t + c_2.$
Since $s(0) = 0$, we have $c_2 = 0$. Thus,
$s(t) = \dfrac{1}{12}t^4 + \dfrac{1}{2}t^2 + 4t.$

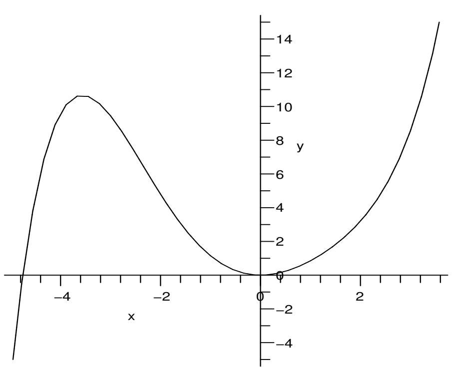
49. (a) There are many correct answers, but any correct answer will be a vertical shift of these answers.



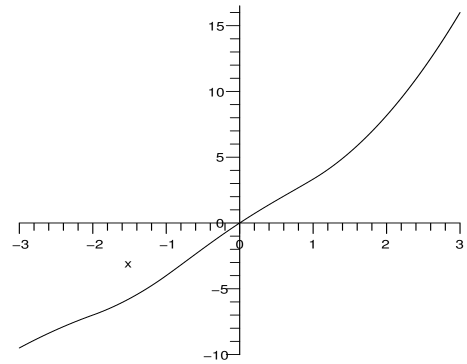
(b) There are many correct answers, but any correct answer will be a vertical shift of these answers.



50. (a) There are many correct answers, but any correct answer will be a vertical shift of these answers.



(b) There are many correct answers, but any correct answer will be a vertical shift of these answers.



51. We start by taking antiderivatives:
$f'(x) = x^2/2 - x + c_1$
$f(x) = x^3/6 - x^2/2 + c_1 x + c_2$.
Now, we use the data that we are given. We know that $f(1) = 2$ and $f'(1) = 3$, which gives us
$3 = f'(1) = 1/2 - 1 + c_1$,
and
$1 = f(1) = 1/6 - 1/2 + c_1 + c_2$.
Therefore $c_1 = 7/2$ and $c_2 = -13/6$ and the function is
$f(x) = \dfrac{x^3}{6} - \dfrac{x^2}{2} + \dfrac{7x}{2} - \dfrac{13}{6}$.

52. We start by taking antiderivatives:
$f'(x) = 3x^2 + 4x + c_1$
$f(x) = x^3 + 2x^2 + c_1 x + c_2$.
Now, we use the data that we are given. We know that $f(-1) = 1$ and $f'(-1) = 2$, which gives us
$2 = f'(-1) = -1 + c_1$,
and
$1 = f(-1) = 1 - c_1 + c_2$.
Therefore $c_1 = 3$ and $c_2 = 3$ and the function is
$f(x) = x^3 + 2x^2 + 3x - 3$.

53. $\dfrac{d}{dx}\left[\sin x^2\right] = 2x\cos x^2$
Therefore,
$\displaystyle\int 2x\cos x^2 dx = \sin x^2 + c$

54. $\dfrac{d}{dx}\left[(x^3+2)^{3/2}\right] = \dfrac{9}{2}x^2(x^3+2)^{1/2}$
Therefore,
$\displaystyle\int x^2\sqrt{x^3+2}dx = \frac{2}{9}(x^3+2)^{3/2} + c$

55. $\dfrac{d}{dx}\left[x^2\sin 2x\right] = 2(x\sin 2x + x^2\cos 2x)$
Therefore,

$$\int \left(x\sin 2x + x^2\cos 2x\right)dx$$
$$= \frac{1}{2}x^2\sin 2x + c$$

56. $\dfrac{d}{dx}\dfrac{x^2}{e^{3x}} = \dfrac{2xe^{3x} - 3x^2e^{3x}}{e^{6x}}$

Therefore,
$$\int \frac{2xe^{3x} - 3x^2e^{3x}}{e^{6x}}dx = \frac{x^2}{e^{3x}} + c$$

57. $\displaystyle\int \frac{x\cos(x^2)}{\sqrt{\sin(x^2)}}dx = \sqrt{\sin(x^2)} + c$

58. $\dfrac{d}{dx}\left(2\sqrt{x}\sin x\right) = 2\sqrt{x}\cos x + \dfrac{1}{\sqrt{x}}\sin x$
$$\int \left(2\sqrt{x}\cos x + \frac{1}{\sqrt{x}}\sin x\right)dx$$
$$= 2\sqrt{x}\sin x + c$$

59. Use a CAS to find antiderivatives and verify by computing the derivatives:

For 11.1(b):
$$\int \sec x dx = \ln|\sec x + \tan x| + c$$
Verify:
$$\frac{d}{dx}\ln|\sec x + \tan x|$$
$$= \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x} = \sec x$$
For 11.1(f):
$$\int x\sin 2x dx = \frac{\sin 2x}{4} - \frac{x\cos 2x}{2} + c$$
Verify:
$$\frac{d}{dx}\left(\frac{\sin 2x}{4} - \frac{x\cos 2x}{2}\right)$$
$$= \frac{2\cos 2x}{4} - \frac{\cos 2x - 2x\sin 2x}{2}$$
$$= x\sin 2x$$

60. Use a CAS to find antiderivatives and verify by computing the derivatives:

For 31(a): The answer is too complicated to be presented here.

For 32(b): $\dfrac{1}{9}\left(3x + \sqrt{3}\ln\dfrac{2\sqrt{3} - 3x}{2\sqrt{3} + 3x}\right) + c$

Verify:
$$\frac{d}{dx}\left[\frac{1}{9}\left(3x + \sqrt{3}\ln\frac{2\sqrt{3} - 3x}{2\sqrt{3} + 3x}\right)\right]$$
$$= \frac{1}{9}\left(3 + \frac{2\sqrt{3} + 3x}{2\sqrt{3} - 3x}\cdot \frac{-3(2\sqrt{3} + 3x) - 3(2\sqrt{3} - 3x)}{(2\sqrt{3} + 3x)^2}\right)$$
$$= \frac{1}{9}\left(3 - \frac{36}{12 - 9x^2}\right) = \frac{x^2}{3x^2 - 4}$$
For 33(a): Almost the same as in Exercise 59, example 1.11 (b).

For 34(b): $\dfrac{1}{2}\ln\dfrac{x-1}{x+1} + c$

Verify:
$$\frac{d}{dx}\left(\frac{1}{2}\ln\frac{x-1}{x+1}\right)$$
$$= \frac{1}{2}\cdot\frac{x+1}{x-1}\cdot\frac{(x+1) - (x-1)}{(x+1)^2}$$
$$= \frac{1}{x^2 - 1}$$

61. Use a CAS to find antiderivatives and verify by computing the derivatives:

(a) $\displaystyle\int x^2e^{-x^3}dx = -\frac{1}{3}e^{-x^3} + c$

Verify:
$$\frac{d}{dx}\left(-\frac{1}{3}e^{-x^3}\right)$$
$$= -\frac{1}{3}e^{-x^3}\cdot(-3x^2)$$
$$= x^2e^{-x^3}$$

(b) $\displaystyle\int \frac{1}{x^2 - x}dx = \ln|x-1| - \ln|x| + c$ Verify:
$$\frac{d}{dx}\left(\ln|x-1| - \ln|x|\right)$$
$$= \frac{1}{x-1} - \frac{1}{x} = \frac{x - (x-1)}{x(x-1)}$$
$$= \frac{1}{x(x-1)} = \frac{1}{x^2 - x}$$

(c) $\displaystyle\int \sec x dx = \ln|\sec x + \tan x| + c$

Verify:
$$\frac{d}{dx}\left[\ln|\sec x + \tan x|\right]$$
$$= \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x}$$
$$= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} = \sec x$$

62. Use a CAS to find antiderivatives and verify by computing the derivatives:

(a) $\displaystyle\int \frac{x}{x^4 + 1}dx = \frac{1}{2}\arctan x^2 + c$

Verify:
$$\frac{d}{dx}\left(\frac{1}{2}\arctan x^2\right)$$
$$= \frac{1}{2}\cdot\frac{1}{x^4+1}\cdot 2x = \frac{x}{x^4+1}$$

(b) $\displaystyle\int 3x\sin 2x dx$
$$= \frac{3}{4}\sin 2x - \frac{3x}{2}\cos 2x + c$$

Verify:
$\dfrac{d}{dx}\left(\dfrac{3}{4}\sin 2x - \dfrac{3x}{2}\cos 2x\right)$
$= \dfrac{3}{2}\cos 2x - \dfrac{3}{2}\cos 2x + 3x\sin 2x$
$= 3x\sin 2x$

(c) $\displaystyle\int \ln x dx = x\ln x - x + c$
Verify:
$\dfrac{d}{dx}(x\ln x - x) = \ln x + 1 - 1$
$= \ln x$

63. $\displaystyle\int \frac{-1}{\sqrt{1-x^2}}dx = \cos^{-1}(x) + c_1$
$\displaystyle\int \frac{-1}{\sqrt{1-x^2}}dx = -\sin^{-1}(x) + c_2$
Therefore,
$\cos^{-1} x + c_1 = -\sin^{-1} x + c_2$
Therefore,
$\sin^{-1} x + \cos^{-1} x =$ constant
To find the value of the constant, let x be any convenient value.
Suppose $x = 0$; then $\sin^{-1} 0 = 0$ and $\cos^{-1} 0 = \pi/2$, so
$\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$

64. To derive these formulas, all that needs to be done is to take the derivatives to see that the integrals are correct:
$\dfrac{d}{dx}(\tan x) = \sec^2 x$
$\dfrac{d}{dx}(\sec x) = \sec x \tan x$

65. To derive these formulas, all that needs to be done is to take the derivatives to see that the integrals are correct:
$\dfrac{d}{dx}(e^x) = e^x$
$\dfrac{d}{dx}(-e^{-x}) = e^{-x}$

66. (a) $\displaystyle\int \frac{1}{kx}dx = \frac{1}{k}\int \frac{1}{x}dx$
$= \dfrac{1}{k}\ln|x| + c_1$

(b) $\displaystyle\int \frac{1}{kx}dx = \frac{1}{k}\int \frac{k}{kx}dx$
$= \dfrac{1}{k}\ln|kx| + c_2$

Because
$\dfrac{1}{k}\ln|kx| = \dfrac{1}{k}(\ln|k| + \ln|x|)$
$= \dfrac{1}{k}\ln|x| + \dfrac{1}{k}\ln|k| = \dfrac{1}{k}\ln|x| + c$
The two antiderivatives are both correct.

67. The key is to find the velocity and position functions. We start with constant acceleration a, a constant. Then, $v(t) = at + v_0$ where v_0 is the initial velocity. The initial velocity is 30 miles per hour, but since our time is in seconds, it is probably best to work in feet per second (30mph = 44ft/s). $v(t) = at + 44$.
We know that the car accelerates to 50 mph (50mph = 73ft/s) in 4 seconds, so $v(4) = 73$.
Therefore, $a \cdot 4 + 44 = 73$ and $a = \dfrac{29}{4}$ ft/s
So,
$v(t) = \dfrac{29}{4}t + 44$ and
$s(t) = \dfrac{29}{8}t^2 + 44t + s_0$
where s_0 is the initial position. We can assume the the starting position is $s_0 = 0$.
Then, $s(t) = \dfrac{29}{8}t^2 + 44t$ and the distance traveled by the car during the 4 seconds is $s(4) = 234$ feet.

68. The key is to find the velocity and position functions. We start with constant acceleration a, a constant. Then, $v(t) = at + v_0$ where v_0 is the initial velocity. The initial velocity is 60 miles per hour, but since our time is in seconds, it is probably best to work in feet per second (60mph = 88ft/s). $v(t) = at + 88$.
We know that the car comes to rest in 3 seconds, so $v(3) = 0$.
Therefore,
$a(3) + 88 = 0$ and $a = -88/3$ft/s (the acceleration should be negative since the car is actually decelerating.
So,
$v(t) = -\dfrac{88}{3}t + 88$ and
$s(t) = -\dfrac{44}{3}t^2 + 88t + s_0$ where s_0 is the initial position. We can assume the the starting position is $s_0 = 0$.
Then, $s(t) = -\dfrac{44}{3}t^2 + 88t$ and the stopping distance is $s(3) = 132$ feet.

69. To estimate the acceleration over each interval, we estimate $v'(t)$ by computing the slope of the tangent lines. For example, for the interval $[0, 0.5]$:
$a \approx \dfrac{v(0.5) - v(0)}{0.5 - 0} = -31.6$ m/s^2.
Notice, acceleration should be negative since the object is falling.
To estimate the distance traveled over the interval, we estimate the velocity and multiply by the time (distance is rate times time). For

an estimate for the velocity, we will use the average of the velocities at the endpoints. For example, for the interval $[0, 0.5]$, the time interval is 0.5 and the velocity is -11.9. Therefore the position changed is $(-11.9)(0.5) = -5.95$ meters. The distance traveled will be 5.95 meters (distance should be positive).

Interval	Accel	Dist
$[0.0, 0.5]$	-31.6	5.95
$[0.5, 1.0]$	-2	12.925
$[1.0, 1.5]$	-11.6	17.4
$[1.5, 2.0]$	-3.6	19.3

70. To estimate the acceleration over each interval, we estimate $v'(t)$ by computing the slope of the tangent lines. For example, for the interval $[0, 1.0]$:
$a \approx \dfrac{v(1.0) - v(0)}{1.0 - 0} = -9.8 \text{ m/s}^2$.
Notice, acceleration should be negative since the object is falling.
To estimate the distance traveled over the interval, we estimate the velocity and multiply by the time (distance is rate times time). For an estimate for the velocity, we will use the average of the velocities at the endpoints. For example, for the interval $[0, 1.0]$, the time interval is 1.0 and the velocity is -4.9. Therefore the position changed is $(-4.9)(1.0) = -4.9$ meters. The distance traveled will be 4.9 meters (distance should be positive).

Interval	Accel	Dist
$[0.0, 1.0]$	-9.8	4.9
$[1.0, 2.0]$	-8.8	14.2
$[2.0, 3.0]$	-6.3	21.75
$[3.0, 4.0]$	-3.6	26.7

71. To estimate the speed over the interval, we first approximate the acceleration over the interval by averaging the acceleration at the endpoint of the interval. Then, the velocity will be the acceleration times the length of time. The slope of the tangent lines. For example, for the interval $[0, 0.5]$ the average acceleration is -0.9 and $v(0.5) = 70 + (-0.9)(0.5) = 69.55$.
And, the distance traveled is the speed times the length of time. For the time $t = 0.5$, the distance would be $\dfrac{70 + 69.55}{2} \times 0.5 \approx 34.89$ meters.

Time	Speed	Dist
0	70	0
0.5	69.55	34.89
1.0	70.3	69.85
1.5	70.35	105.01
2.0	70.65	104.26

72. To estimate the speed over the interval, we first approximate the acceleration over the interval by averaging the acceleration at the endpoint of the interval. Then, the velocity will be the acceleration times the length of time. the slope of the tangent lines. For example, for the interval $[0.0, 0.5]$ the average acceleration is -0.8 and $v(0.5) = 20 + (-0.8)(.5) = 19.6$. Of course, speed is the absolute value of the velocity.
And, the distance traveled is the average speed times the length of time. For the time $t = 0.5$, the distance would be $\dfrac{20 + 19.6}{2} \times 0.5 = 9.9$ meters.

Time	Speed	Dist
0	20	0
0.5	19.6	9.9
1.0	17.925	19.281
1.5	16.5	27.888
2.0	16.125	34.044

4.2 Sums And Sigma Notation

1. The given sum is the sum of twice the squares of the integers from 1 to 14.
$$2(1)^2 + 2(2)^2 + 2(3)^2 + \ldots + 2(14)^2 = \sum_{i=1}^{14} 2i^2$$

2. The given sum is the sum of squares roots of the integers from 1 to 14.
$$\sqrt{2-1} + \sqrt{3-1} + \sqrt{4-1} + \ldots + \sqrt{15-1}$$
$$= \sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{13} + \sqrt{14}$$
$$= \sum_{i=1}^{14} \sqrt{i}$$

3. (a) $\displaystyle\sum_{i=1}^{50} i^2 = \frac{(50)(51)(101)}{6} = 42,925$

(b) $\displaystyle\left(\sum_{i=1}^{50} i\right)^2 = \left(\frac{50(51)}{2}\right)^2 = 1,625,625$

4. (a) $\displaystyle\sum_{i=1}^{10} \sqrt{i}$
$= 1 + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6}$

$$+\sqrt{7}+\sqrt{8}+\sqrt{9}+\sqrt{10}$$
$$\approx 22.47$$

(b) $\sqrt{\sum_{i=1}^{10} i} = \sqrt{\frac{10(11)}{2}} = \sqrt{55}$

5. $\sum_{i=1}^{6} 3i^2 = 3+12+27+48+75+108$
$= 273$

6. $\sum_{i=3}^{7} i^2+i = 12+20+30+42+56$
$= 160$

7. $\sum_{i=6}^{10}(4i+2)$
$= (4(6)+2)+(4(7)+2)+(4(8)+2)$
$+ (4(9)+2)+(4(10)+2)$
$= 26+30+34+38+42$
$= 170$

8. $\sum_{i=6}^{8}(i^2+2)$
$= (6^2+2)+(7^2+2)+(8^2+2)$
$= 38+51+66 = 155$

9. $\sum_{i=1}^{70}(3i-1) = 3\cdot\sum_{i=1}^{70} i - 70$
$= 3\cdot\frac{70(71)}{2} - 70 = 7,385$

10. $\sum_{i=1}^{45}(3i-4) = 3\sum_{i=1}^{45} i - 4\sum_{i=1}^{45} 1$
$= 3\left(\frac{45(46)}{2}\right) - 4(45) = 2925$

11. $\sum_{i=1}^{40}(4-i^2) = 160 - \sum_{i=1}^{40} i^2$
$= 160 - \frac{(40)(41)(81)}{6}$
$= 160 - 22,140 = -21,980$

12. $\sum_{i=1}^{50}(8-i) = 8\sum_{i=1}^{50} 1 - \sum_{i=1}^{50} i$
$= 8(50) - \frac{50(51)}{2} = -875$

13. $\sum_{n=1}^{100}\left(n^2-3n+2\right)$
$= \sum_{n=1}^{100} n^2 - 3\sum_{n=1}^{100} n + \sum_{n=1}^{100} 2$
$= \frac{(100)(101)(201)}{6} - 3\frac{100(101)}{2} + 200$
$= 338,350 - 15,150 + 200 = 323,400$

14. $\sum_{n=1}^{140}\left(n^2+2n-4\right)$
$= \sum_{n=1}^{140} n^2 + 2\sum_{n=1}^{140} n - \sum_{n=1}^{140} 4$
$= \frac{(140)(141)(281)}{6} + 2\left(\frac{140(141)}{2}\right) - 4\,(140)$
$= 943,670$

15. $\sum_{i=3}^{30}\left[(i-3)^2+i-3\right]$
$= \sum_{i=3}^{30}(i-3)^2 + \sum_{i=3}^{30}(i-3)$
$= \sum_{n=0}^{27} n^2 + \sum_{n=0}^{27} n$ (substitute $i-3=n$)
$= 0 + \sum_{n=1}^{27} n^2 + 0 + \sum_{n=1}^{27} n$
$= \frac{27\,(28)\,(55)}{6} + \frac{27\,(28)}{2} = 7308$

16. $\sum_{i=4}^{20}(i-3)\,(i+3) = \sum_{i=4}^{20}\left(i^2-9\right)$
$= \sum_{i=4}^{20} i^2 - 9\sum_{i=4}^{20} 1$
$= \sum_{i=1}^{20} i^2 - \sum_{i=1}^{3} i^2 - 9\sum_{i=4}^{20} 1$
$= \frac{20\,(21)\,(41)}{6} - 1 - 4 - 9 - 9\,(17)$
$= 2703$

17. $\sum_{k=3}^{n}\left(k^2-3\right)$
$= \sum_{k=3}^{n} k^2 + \sum_{k=3}^{n}(-3)$
$= \sum_{k=1}^{n} k^2 - \sum_{k=1}^{2} k^2$
$+ \sum_{k=1}^{n}(-3) - \sum_{k=1}^{2}(-3)$
$= \frac{n\,(n+1)\,(2n+1)}{6} - 1 - 4$
$+ (-3)\,n - (-3)\,(2)$
$= \frac{n\,(n+1)\,(2n+1)}{6} - 5 - 3n + 6$

$$= \frac{n(n+1)(2n+1)}{6} - 3n + 1$$

18. $\displaystyle\sum_{k=0}^{n} \left(k^2+5\right)$

$$= \sum_{k=0}^{n} k^2 + \sum_{k=0}^{n} 5$$
$$= 0 + \sum_{k=1}^{n} k^2 + 5 + \sum_{k=1}^{n} 5$$
$$= \frac{n(n+1)(2n+1)}{6} + 5 + 5n$$

19. $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x$

$$= \sum_{i=1}^{5} (x_i^2 + 4x_i) \cdot 0.2$$
$$= (0.2^2 + 4(0.2))(0.2) + \ldots$$
$$+ (1^2+4)(0.2)$$
$$= (0.84)(0.2) + (1.76)(0.2)$$
$$+ (2.76)(0.2) + (3.84)(0.2)$$
$$+ (5)(0.2)$$
$$= 2.84$$

20. $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x$

$$= \sum_{i=1}^{5} (3x_i + 5) \cdot 0.4$$
$$= (3(0.4)+5)(0.4) + \ldots$$
$$+ (3(2)+5)(0.4)$$
$$= (6.2)(0.4) + (7.4)(0.4)$$
$$+ (8.6)(0.4) + (9.8)(0.4)$$
$$+ (11)(0.4)$$
$$= 17.2$$

21. $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x$

$$= \sum_{i=1}^{10} (4x_i^2 - 2) \cdot 0.1$$
$$= (4(2.1)^2 - 2)(0.1) + \ldots$$
$$+ (4(3)^2 - 2)(0.1)$$
$$= (15.64)(0.1) + (17.36)(0.1)$$
$$+ (19.16)(0.1) + (21.04)(0.1)$$
$$+ (23)(0.1) + (25.04)(0.1)$$
$$+ (27.16)(0.1) + (29.36)(0.1)$$
$$+ (31.64)(0.1) + (34)(0.1)$$
$$= 24.34$$

22. $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x$

$$= \sum_{i=1}^{10} (x^3 + 4) \cdot 0.1$$
$$= ((2.05)^3 + 4)(0.1) + \ldots$$
$$+ ((2.95)^3 + 4)(0.1)$$
$$= (202.4375)(0.1)$$
$$= 20.24375$$

23. $\displaystyle\sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^2 + 2\left(\frac{i}{n}\right)\right]$

$$= \frac{1}{n}\left[\sum_{i=1}^{n} \frac{i^2}{n^2} + 2\sum_{i=1}^{n} \frac{i}{n}\right]$$
$$= \frac{1}{n}\left[\frac{1}{n^2}\sum_{i=1}^{n} i^2 + \frac{2}{n}\sum_{i=1}^{n} i\right]$$
$$= \frac{1}{n}\left[\frac{1}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right)\right.$$
$$\left. + \frac{2}{n}\left(\frac{n(n+1)}{2}\right)\right]$$
$$= \frac{n(n+1)(2n+1)}{6n^3} + \frac{n(n+1)}{n^2}$$
$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^2 + 2\left(\frac{i}{n}\right)\right]$$
$$= \lim_{n\to\infty}\left[\frac{n(n+1)(2n+1)}{6n^3} + \frac{n(n+1)}{n^2}\right]$$
$$= \frac{2}{6} + 1 = \frac{4}{3}$$

24. $\displaystyle\sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^2 - 5\left(\frac{i}{n}\right)\right]$

$$= \frac{1}{n}\left[\sum_{i=1}^{n} \frac{i^2}{n^2} - 5\sum_{i=1}^{n} \frac{i}{n}\right]$$
$$= \frac{1}{n}\left[\frac{1}{n^2}\sum_{i=1}^{n} i^2 - \frac{5}{n}\sum_{i=1}^{n} i\right]$$
$$= \frac{1}{n}\left[\frac{1}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right)\right.$$
$$\left. - \frac{5}{n}\left(\frac{n(n+1)}{2}\right)\right]$$
$$= \frac{n(n+1)(2n+1)}{6n^3} - \frac{5n(n+1)}{2n^2}$$
$$= \frac{-13n^2 - 12n + 1}{6n^2}$$
$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{i}{n}\right)^2 - 5\left(\frac{i}{n}\right)\right]$$
$$= \lim_{n\to\infty} \frac{-13n^2 - 12n + 1}{6n^2}$$
$$= \lim_{n\to\infty} -\frac{13}{6} - \frac{12}{6n} + \frac{1}{6n^2}$$

$= -\dfrac{13}{6}$

25. $\displaystyle\sum_{i=1}^{n} \frac{1}{n}\left[4\left(\frac{2i}{n}\right)^2 - \left(\frac{2i}{n}\right)\right]$

$$= \frac{1}{n}\left[16\sum_{i=1}^{n}\frac{i^2}{n^2} - 2\sum_{i=1}^{n}\frac{i}{n}\right]$$

$$= \frac{1}{n}\left[\frac{16}{n^2}\sum_{i=1}^{n} i^2 - \frac{2}{n}\sum_{i=1}^{n} i\right]$$

$$= \frac{1}{n}\left[\frac{16}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{2}{n}\left(\frac{n(n+1)}{2}\right)\right]$$

$$= \frac{16n(n+1)(2n+1)}{6n^3} - \frac{n(n+1)}{n^2}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left[4\left(\frac{2i}{n}\right)^2 - \left(\frac{2i}{n}\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{16n(n+1)(2n+1)}{6n^3} - \frac{n(n+1)}{n^2}\right]$$

$$= \frac{16}{3} - 1 = \frac{13}{3}$$

26. $\displaystyle\sum_{i=1}^{n} \frac{1}{n}\left[\left(\frac{2i}{n}\right)^2 + 4\left(\frac{i}{n}\right)\right]$

$$= \frac{1}{n}\left[\sum_{i=1}^{n}\frac{4i^2}{n^2} + 4\sum_{i=1}^{n}\frac{i}{n}\right]$$

$$= \frac{1}{n}\left[\frac{4}{n^2}\sum_{i=1}^{n} i^2 + \frac{4}{n}\sum_{i=1}^{n} i\right]$$

$$= \frac{1}{n}\left[\frac{4}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{4}{n}\left(\frac{n(n+1)}{2}\right)\right]$$

$$= \frac{4n(n+1)(2n+1)}{6n^3} + \frac{4n(n+1)}{2n^2}$$

$$= \frac{10n^2+12n+2}{3n^2}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left[\left(\frac{2i}{n}\right)^2 + 4\left(\frac{i}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{10n^2+12n+2}{3n^2}$$

$$= \lim_{n\to\infty}\frac{10}{3} + \frac{12}{3n} + \frac{2}{3n^2} = \frac{10}{3}$$

27. Want to prove that

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

is true for all integers $n \geq 1$.

For $n = 1$, we have

$$\sum_{i=1}^{1} i^3 = 1 = \frac{1^2(1+1)^2}{4},$$

as desired.

So the proposition is true for $n = 1$.

Next, assume that

$$\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4},$$

for some integer $k \geq 1$.

In this case, we have by the induction assumption that for $n = k+1$,

$$\sum_{i=1}^{n} i^3 = \sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

$$= \frac{(k+1)^2(k^2+4k+4)}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

$$= \frac{n^2(n+1)^2}{4}$$

as desired.

28. Want to prove that

$$\sum_{i=1}^{n} i^5 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12}$$

is true for all integers $n \geq 1$.

For $n = 1$, we have

$$\sum_{i=1}^{1} i^3 = 1 = \frac{1^2(1+1)^2(2+2-1)}{12},$$

as desired.

So the proposition is true for $n = 1$.

Next, assume that

$$\sum_{i=1}^{k} i^5 = \frac{k^2(k+1)^2(2k^2+2k-1)}{12},$$

for some integer $k \geq 1$.

In this case, we have by the induction assumption that for $n = k+1$,

$$\sum_{i=1}^{n} i^5 = \sum_{i=1}^{k+1} i^5 = \sum_{i=1}^{k} i^5 + (k+1)^5$$

$$= \frac{k^2(k+1)^2(2k^2+2k-1)}{12} + (k+1)^5$$

$$= \frac{k^2(k+1)^2(2k^2+2k-1) + 12(k+1)^5}{12}$$

$$= \frac{(k+1)^2[k^2(2k^2+2k-1) + 12(k+1)^3]}{12}$$

$= \dfrac{(k+1)^2[2k^4+14k^3+35k^2+36k+12]}{12}$
$= \dfrac{(k+1)^2(k^2+4k+4)(2k^2+6k+3)}{12}$
$= \dfrac{n^2(n+1)^2(2n^2+2n-1)}{12}$
as desired.

29. $\displaystyle\sum_{i=1}^{10}(i^3-3i+1)$
$\displaystyle= \sum_{i=1}^{10} i^3 - 3\sum_{i=1}^{10} i + 10$
$= \dfrac{100(11)^2}{4} - 3\dfrac{10(11)}{2} + 10$
$= 2,870$

30. $\displaystyle\sum_{i=1}^{20}(i^3+2i)$
$\displaystyle= \sum_{i=1}^{20} i^3 + 2\sum_{i=1}^{20} i$
$= \dfrac{400(21)^2}{4} + 2\dfrac{20(21)}{2} = 44,520$

31. $\displaystyle\sum_{i=1}^{100}(i^5-2i^2)$
$\displaystyle= \sum_{i=1}^{100} i^5 - 2\sum_{i=1}^{100} i^2$
$= \dfrac{(100^2)(101^2)[2(100^2)+2(100)-1]}{12}$
$- 2\dfrac{100(101)(201)}{6}$
$= 171,707,655,800$

32. $\displaystyle\sum_{i=1}^{100}(2i^5+2i+1)$
$\displaystyle= 2\sum_{i=1}^{100} i^5 + 2\sum_{i=1}^{100} i + 100$
$= 2\dfrac{(100^2)(101^2)[2(100^2)+2(100)-1]}{12}$
$+ 2\cdot\dfrac{100(101)}{2} + 100$
$= 343,416,675,200$

33. $\displaystyle\sum_{i=1}^{n}(ca_i+db_i) = \sum_{i=1}^{n} ca_i + \sum_{i=1}^{n} db_i$
$\displaystyle= c\sum_{i=1}^{n} a_i + d\sum_{i=1}^{n} b_i$

34. When $n=0$, $a = \dfrac{a-ar}{1-r}$.
Assume the formula holds for $n=k-1$, which gives
$a + ar + \cdots ar^{k-1} = \dfrac{a-ar^k}{1-r}$.
Then for $n=k$,
we have $a + ar + \cdots ar^k$
$= a + ar + \cdots ar^{k-1} + ar^k$
$= \dfrac{a-ar^k}{1-r} + ar^k$
$= \dfrac{a-ar^k+ar^k(1-r)}{1-r}$
$= \dfrac{a-ar^k+ar^k-ar^{k+1}}{1-r}$
$= \dfrac{a-ar^{k+1}}{1-r}$
$= \dfrac{a-ar^{n+1}}{1-r}$
as desired.

35. $\displaystyle\sum_{i=1}^{n} e^{6i/n}\left(\frac{6}{n}\right)$
$\displaystyle= \frac{6}{n}\sum_{i=1}^{n} e^{6i/n}$
$= \dfrac{6}{n}\left(\dfrac{e^{6/n}-e^6}{1-e^{6/n}}\right)$
$= \dfrac{6}{n}\left(\dfrac{1-e^6}{1-e^{6/n}} - 1\right)$
$= \dfrac{6}{n}\dfrac{1-e^6}{1-e^{6/n}} - \dfrac{6}{n}$
Now $\displaystyle\lim_{x\to\infty}\frac{6}{n} = 0$, and
$\displaystyle\lim_{x\to\infty}\frac{6}{n}\frac{1-e^6}{1-e^{6/n}}$
$\displaystyle= 6(1-e^6)\lim_{x\to\infty}\frac{1/n}{1-e^{6/n}}$
$\displaystyle= 6(1-e^6)\lim_{x\to\infty}\frac{1}{-6e^{6/n}}$
$= e^6 - 1.$
Thus $\displaystyle\lim_{x\to\infty}\sum_{i=1}^{n} e^{6i/n}\frac{6}{n} = e^6 - 1.$

36. $\displaystyle\sum_{i=1}^{n} e^{(2i)/n}\frac{2}{n}$
$= \dfrac{2}{n}\left(\dfrac{e^{2/n}-e^2}{1-e^{2/n}}\right)$
$= \dfrac{2}{n}\left(\dfrac{1-e^2}{1-e^{2/n}} - 1\right)$

$= \frac{2}{n}\frac{1-e^2}{1-e^{2/n}} - \frac{2}{n}$

Now $\lim_{x\to\infty} \frac{2}{n} = 0$, and

$\lim_{x\to\infty} \frac{2}{n}\frac{1-e^2}{1-e^{2/n}}$

$= 2(1-e^2)\lim_{x\to\infty} \frac{1/n}{1-e^{2/n}}$

$= 2(1-e^2)\lim_{x\to\infty} \frac{1}{-2e^{2/n}}$

$= e^2 - 1.$

Thus $\lim_{x\to\infty} \sum_{i=1}^{n} e^{2i/n}\frac{2}{n} = e^2 - 1.$

37. Distance
$= 50(2) + 60(1) + 70(1/2) + 60(3)$
$= 375$ miles.

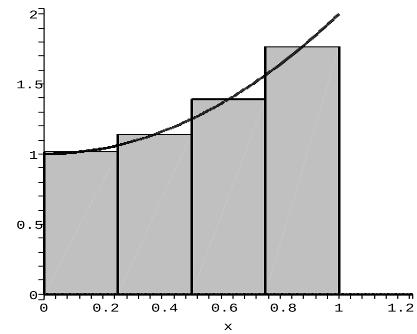
38. Distance
$= 50(1) + 40(1) + 60(1/2) + 55(3)$
$= 285$ miles.

39. On the time interval $[0, 0.25]$, the estimated velocity is the average velocity $\frac{120+116}{2} = 118$ feet per second.
We estimate the distance traveled during the time interval $[0, 0.25]$ to be
$(118)(0.25 - 0) = 29.5$ feet.
Altogether, the distance traveled is estimated as
$= (236/2)(0.25) + (229/2)(0.25)$
$+ (223/2)(0.25) + (218/2)(0.25)$
$+ (214/2)(0.25) + (210/2)(0.25)$
$+ (207/2)(0.25) + (205/2)(0.25)$
$= 217.75$ feet.

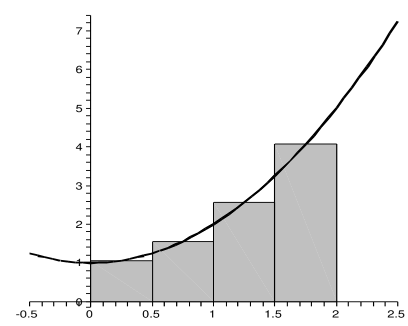
40. On the time interval $[0, 0.5]$, the estimated velocity is the average velocity $\frac{10+14.9}{2} = 12.45$ meters per second. We estimate the distance fallen during the time interval $[0, 0.5]$ to be
$(12.45)(0.5 - 0) = 6.225$ meters.
Altogether, the distance fallen (estimated)
$= (12.45)(0.5) + (17.35)(0.5)$
$+ (22.25)(0.5) + (27.15)(0.5)$
$+ (32.05)(0.5) + (36.95)(0.5)$
$+ (41.85)(0.5) + (46.75)(0.5)$
$= 118.4$ meters.

4.3 Area

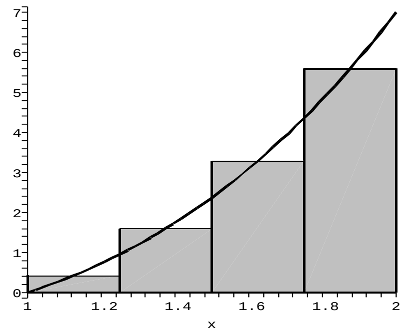
1. (a) Evaluation points:
0.125, 0.375, 0.625, 0.875.
Notice that $\Delta x = 0.25$.
$A_4 = [f(0.125) + f(0.375) + f(0.625)$
$+ f(0.875)](0.25)$
$= [(0.125)^2 + 1 + (0.375)^2 + 1$
$+ (0.625)^2 + 1 + (0.875)^2 + 1](0.25)$
$= 1.38125.$



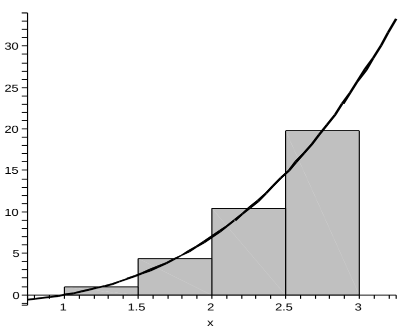
(b) Evaluation points:
0.25, 0.75, 1.25, 1.75.
Notice that $\Delta x = 0.5$.
$A_4 = [f(0.25) + f(0.75) + f(1.25)$
$+ f(1.75)](0.5)$
$= [(0.25)^2 + 1 + (0.75)^2 + 1 + (1.25)^2$
$+ 1 + (1.75)^2 + 1](0.5)$
$= 4.625.$



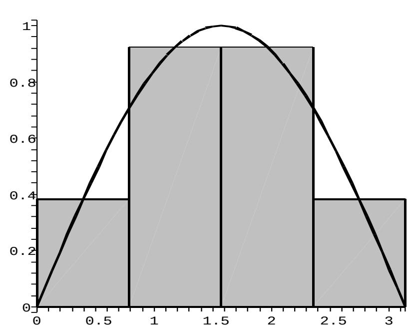
2. (a) Evaluation points:
1.125, 1.375, 1.625, 1.875.
Notice that $\Delta x = 0.25$.
$A_4 = [f(1.125) + f(1.375) + f(1.625)$
$+ f(1.875)](0.25)$
$= [(1.125)^3 - 1 + (1.375)^3 - 1$
$+ (1.625)^3 - 1 + (1.875)^3 - 1](0.25)$
$= 2.7265625.$



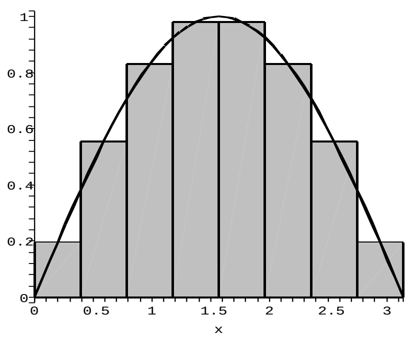
(b) Evaluation points:
1.25, 1.75, 2.25, 2.75.
Notice that $\Delta x = 0.5$.
$A_4 = [f(1.25) + f(1.75) + f(2.25) + f(2.75)](0.5)$
$= [(1.25)^3 - 1 + (1.75)^3 - 1 + (2.25)^3 - 1 + (2.75)^3 - 1](0.5)$
$= 17.75$.



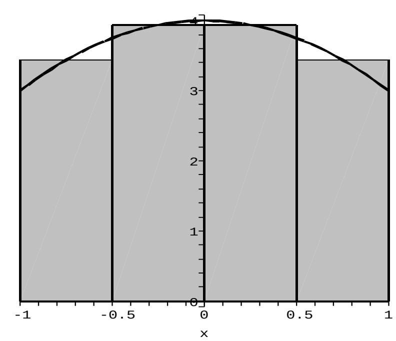
3. (a) Evaluation points:
$\pi/8$, $3\pi/8$, $5\pi/8$, $7\pi/8$.
Notice that $\Delta x = \pi/4$.
$A_4 = [f(\pi/8) + f(3\pi/8) + f(5\pi/8) + f(7\pi/8)](\pi/4)$
$= [\sin(\pi/8) + \sin(3\pi/8) + \sin(5\pi/8) + \sin(7\pi/8)](\pi/4)$
$= 2.05234$.



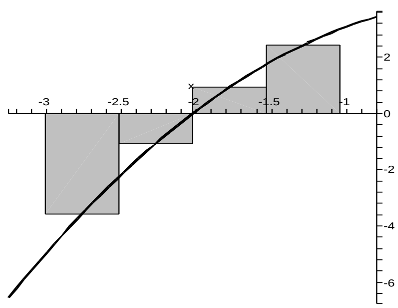
(b) Evaluation points:
$\pi/16$, $3\pi/16$, $5\pi/16$, $7\pi/16$, $9\pi/16$, $11\pi/16$, $13\pi/16$, $15\pi/16$.
Notice that $\Delta x = \pi/8$.
$A_4 = [f(\pi/16) + f(3\pi/16) + f(5\pi/16) + f(7\pi/16) + f(9\pi/16) + f(11\pi/16) + f(13\pi/16) + f(15\pi/16)](\pi/8)$
$= [\sin(\pi/16) + \sin(3\pi/16) + \sin(5\pi/16) + \sin(7\pi/16) + \sin(9\pi/16) + \sin(11\pi/16) + \sin(13\pi/16) + \sin(15\pi/16)](\pi/8)$
$= 2.0129$.



4. (a) Evaluation points:
-0.75, -0.25, 0.25, 0.75.
Notice that $\Delta x = 0.5$.
$A_4 = [f(-0.75) + f(-0.25) + f(0.25) + f(0.75)](0.5)$
$= [4 - (-0.75)^2 + 4 - (-0.25)^2 + 4 - (0.25)^2 + 4 - (0.75)^2](0.5)$
$= 7.375$.



(b) Evaluation points:
-2.75, -2.25, -1.75, -1.25.
Notice that $\Delta x = 0.5$.
$A_4 = [f(-2.75) + f(-2.25) + f(-1.75) + f(-1.25)](0.5)$
$= [4 - (-2.75)^2 + 4 - (-2.25)^2 + 4 - (-1.75)^2 + 4 - (-1.25)^2](0.5)$
$= -0.625$.



5. (a) There are 16 rectangles and the evaluation points are given by $c_i = i\Delta x$ where i is from 0 to 15.

$$A_{16} = \Delta x \sum_{i=0}^{15} f(c_i)$$
$$= \frac{1}{16}\sum_{i=0}^{15}\left[\left(\frac{i}{16}\right)^2 + 1\right] \approx 1.3027$$

(b) There are 16 rectangles and the evaluation points are given by $c_i = i\Delta x + \dfrac{\Delta x}{2}$ where i is from 0 to 15.

$$A_{16} = \Delta x \sum_{i=0}^{15} f(c_i)$$
$$= \frac{1}{16}\sum_{i=0}^{15}\left[\left(\frac{i}{16} + \frac{1}{32}\right)^2 + 1\right]$$
$$\approx 1.3330$$

(c) There are 16 rectangles and the evaluation points are given by $c_i = i\Delta x + \Delta x$ where i is from 0 to 15.

$$A_{16} = \Delta x \sum_{i=0}^{15} f(c_i)$$
$$= \frac{1}{16}\sum_{i=0}^{15}\left[\left(\frac{i}{16} + \frac{1}{16}\right)^2 + 1\right]$$
$$\approx 1.3652$$

6. (a) There are 16 rectangles and the evaluation points are given by $c_i = i\Delta x$ where i is from 0 to 15.

$$A_{16} = \Delta x \sum_{i=0}^{15} f(c_i)$$
$$= \frac{1}{8}\sum_{i=0}^{15}\left[\left(\frac{i}{8}\right)^2 + 1\right] \approx 4.4219$$

(b) There are 16 rectangles and the evaluation points are given by $c_i = i\Delta x + \dfrac{\Delta x}{2}$ where i is from 0 to 15.

$$A_{16} = \Delta x \sum_{i=0}^{15} f(c_i)$$
$$= \frac{1}{8}\sum_{i=0}^{15}\left[\left(\frac{i}{8} + \frac{1}{16}\right)^2 + 1\right] \approx 4.6640$$

(c) There are 16 rectangles and the evaluation points are given by $c_i = i\Delta x + \Delta x$ where i is from 0 to 15.

$$A_{16} = \Delta x \sum_{i=0}^{15} f(c_i)$$
$$= \frac{1}{8}\sum_{i=0}^{15}\left[\left(\frac{i}{8} + \frac{1}{8}\right)^2 + 1\right] \approx 4.9219$$

7. (a) There are 16 rectangles and the evaluation points are the left endpoints which are given by
$c_i = 1 + i\Delta x$ where i is from 0 to 15.

$$A_{16} = \Delta x \sum_{i=0}^{15} f(c_i)$$
$$= \frac{3}{16}\sum_{i=0}^{15}\sqrt{1 + \frac{3i}{16} + 2} \approx 6.2663$$

(b) There are 16 rectangles and the evaluation points are the midpoints which are given by
$c_i = 1 + i\Delta x + \dfrac{\Delta x}{2}$ where i is from 0 to 15.

$$A_{16} = \Delta x \sum_{i=0}^{15} f(c_i)$$
$$= \frac{3}{16}\sum_{i=0}^{15}\sqrt{1 + \frac{3i}{16} + \frac{3}{32} + 2}$$
$$\approx 6.3340$$

(c) There are 16 rectangles and the evaluation points are the right endpoints which are given by
$c_i = 1 + i\Delta x$ where i is from 1 to 16.

$$A_{16} = \Delta x \sum_{i=1}^{16} f(c_i)$$
$$= \frac{3}{16}\sum_{i=1}^{16}\sqrt{1 + \frac{3i}{16} + 2} \approx 6.4009$$

8. (a) There are 16 rectangles and the evaluation points are the left endpoints which are given by
$c_i = -1 + i\Delta x - \Delta x$
where i is from 1 to 16.

$$A_{16} = \Delta x \sum_{i=1}^{16} f(c_i)$$

$= \frac{1}{8}\sum_{i=1}^{16} e^{-2(-1+\frac{i}{8}-\frac{1}{8})} \approx 4.0991$

(b) There are 16 rectangles and the evaluation points are the midpoints which are given by
$c_i = -1 + i\Delta x - \frac{\Delta x}{2}$
where i is from 1 to 16.
$A_{16} = \Delta x \sum_{i=1}^{16} f(c_i)$
$= \frac{1}{8}\sum_{i=1}^{16} e^{-2(-1+\frac{i}{8}-\frac{1}{16})} \approx 3.6174$

(c) There are 16 rectangles and the evaluation points are the right endpoints which are given by
$c_i = -1 + i\Delta x$ where i is from 1 to 16.
$A_{16} = \Delta x \sum_{i=1}^{16} f(c_i)$
$= \frac{1}{8}\sum_{i=1}^{16} e^{-2(-1+\frac{i}{8})} \approx 3.1924$

9. (a) There are 50 rectangles and the evaluation points are given by $c_i = i\Delta x$ where i is from 0 to 49.
$A_{50} = \Delta x \sum_{i=0}^{50} f(c_i)$
$= \frac{\pi}{100}\sum_{i=0}^{50} \cos\left(\frac{\pi i}{100}\right) \approx 1.0156$

(b) There are 50 rectangles and the evaluation points are given by $c_i = \frac{\Delta x}{2} + i\Delta x$ where i is from 0 to 49.
$A_{50} = \Delta x \sum_{i=0}^{50} f(c_i)$
$= \frac{\pi}{100}\sum_{i=0}^{50} \cos\left(\frac{\pi}{200} + \frac{\pi i}{100}\right)$
≈ 1.00004

(c) There are 50 rectangles and the evaluation points are given by $c_i = \Delta x + i\Delta x$ where i is from 0 to 49.
$A_{50} = \Delta x \sum_{i=0}^{50} f(c_i)$
$= \frac{\pi}{100}\sum_{i=0}^{50} \cos\left(\frac{\pi}{100} + \frac{\pi i}{100}\right)$
≈ 0.9842

10. (a) There are 100 rectangles and the evaluation points are left endpoints which are given by $c_i = -1 + i\Delta x - \Delta x$ where i is from 1 to 100.
$A_{100} = \Delta x \sum_{i=1}^{100} f(c_i)$
$= \frac{2}{100}\sum_{i=1}^{100} \left[\left(-1 + \frac{2i}{100} - \frac{2}{100}\right)^3 - 1\right]$
≈ -2.02

(b) There are 100 rectangles and the evaluation points are midpoints which are given by $c_i = -1 + i\Delta x - \frac{\Delta x}{2}$ where i is from 1 to 100.
$A_{100} = \Delta x \sum_{i=1}^{100} f(c_i)$
$= \frac{2}{100}\sum_{i=1}^{100} \left[\left(-1 + \frac{2i}{100} - \frac{1}{100}\right)^3 - 1\right]$
$= -2$

(c) There are 100 rectangles and the evaluation points are right endpoints which are given by $c_i = -1 + i\Delta x$ where i is from 1 to 100.
$A_{100} = \Delta x \sum_{i=1}^{100} f(c_i)$
$= \frac{2}{100}\sum_{i=1}^{100} \left[\left(-1 + \frac{2i}{100}\right)^3 - 1\right] \approx -1.98$

11. (a) $\Delta x = \frac{1}{n}$. We will use right endpoints as evaluation points, $x_i = \frac{i}{n}$.
$A_n = \sum_{i=1}^{n} f(x_i)\Delta x$
$= \frac{1}{n}\sum_{i=1}^{n} \left[\left(\frac{i}{n}\right)^2 + 1\right] = \frac{1}{n^3}\sum_{i=1}^{n} i^2 + 1$
$= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + 1$
$= \frac{8n^2+3n+1}{6n^2}$
Now to compute the exact area, we take the limit as $n \to \infty$:
$A = \lim_{n\to\infty} A_n \quad = \lim_{n\to\infty} \frac{8n^2+3n+1}{6n^2}$
$= \lim_{n\to\infty} \frac{8}{6} + \frac{3}{6n} + \frac{1}{6n^2} = \frac{4}{3}$

(b) $\Delta x = \frac{2}{n}$. We will use right endpoints as evaluation points, $x_i = \frac{2i}{n}$.
$A_n = \sum_{i=1}^{n} f(x_i)\Delta x$

$$= \frac{2}{n}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^2+1\right]$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^2+1\right]$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)^2+\frac{2}{n}\sum_{i=1}^{n}1$$
$$= \frac{8}{n^3}\sum_{i=1}^{n}i^2+2$$
$$= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]+2$$
$$= \frac{8}{n^2}\left[\frac{(n+1)(2n+1)}{6}\right]+2$$
$$= \frac{4}{3n^2}\left(2n^2+3n+1\right)+2$$
$$= \frac{14n^2+12n+4}{3n^2}$$

Now, to compute the exact area, we take the limit as $n \to \infty$:

$$A = \lim_{n\to\infty} A_n$$
$$= \lim_{n\to\infty}\frac{14n^2+12n+4}{3n^2}$$
$$= \frac{14}{3}$$

(c) $\Delta x = \dfrac{2}{n}$ We will use right endpoints as evaluation points,$x_i = 1+\dfrac{2i}{n}$.

$$A_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$
$$= \sum_{i=1}^{n}\left({x_i}^2+1\right)\left(\frac{2}{n}\right)$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left(\left(1+\frac{2i}{n}\right)^2+1\right)$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left(2+\frac{4i}{n}+\frac{4i^2}{n^2}\right)$$
$$= 4+\frac{8}{n^2}\sum_{i=1}^{n}i+\frac{8}{n^3}\sum_{i=1}^{n}i^2$$
$$= 4+\frac{8}{n^2}\left(\frac{n(n+1)}{2}\right)+\frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$
$$= 4+\left(\frac{4n+4}{n}\right)+\left[\frac{8n^2+12n+4}{3n^2}\right]$$

Now, to compute the exact area, we take the limit as $n\to\infty$:

$$A = \lim_{n\to\infty} A_n$$
$$= \lim_{n\to\infty}\left(4+\frac{4n+4}{n}+\frac{8n^2+12n+4}{3n^2}\right)$$
$$= 4+4+\frac{8}{3} = \frac{32}{3}$$

12. (a) $\Delta x = \dfrac{1}{n}$. We will use right endpoints as evaluation points, $x_i = \dfrac{i}{n}$.

$$A_n = \sum_{i=1}^{n} f(x_i)\Delta x$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^2+3\left(\frac{i}{n}\right)\right]$$
$$= \frac{1}{n^3}\sum_{i=1}^{n}i^2+\frac{3}{n^2}\sum_{i=1}^{n}i$$
$$= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)+\frac{3}{n^2}\left(\frac{n(n+1)}{2}\right)$$
$$= \frac{11n^2+12n+1}{6n^2}$$

Now to compute the exact area, we take the limit as $n\to\infty$:

$$A = \lim_{n\to\infty} A_n$$
$$= \lim_{n\to\infty}\frac{11n^2+12n+1}{6n^2}$$
$$= \lim_{n\to\infty}\frac{11}{6}+\frac{12}{6n}+\frac{1}{6n^2} = \frac{11}{6}$$

(b) $\Delta x = \dfrac{2}{n}$. We will use right endpoints as evalution points, $x_i = \dfrac{2i}{n}$.

$$A_n = \sum_{i=1}^{n} f(x_i)\Delta x$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^2+3\left(\frac{2i}{n}\right)\right]$$
$$= \frac{8}{n^3}\sum_{i=1}^{n}i^2+\frac{12}{n^2}\sum_{i=1}^{n}i$$
$$= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]+\frac{12}{n^2}\left[\frac{n(n+1)}{2}\right]$$
$$= \left[\frac{\left(8n^2+12n+4\right)}{3n^2}\right]+\left[\frac{6n+6}{n}\right]$$

Now, to compute the exact area, we take the limit as $n\to\infty$: $A = \lim\limits_{n\to\infty} A_n$

$$= \lim_{n\to\infty}\left(\frac{\left(8n^2+12n+4\right)}{3n^2}+\frac{6n+6}{n}\right)$$
$$= \frac{8}{3}+6 = \frac{26}{3}$$

(c) $\Delta x = \dfrac{2}{n}$. We will use right endpoints as evalution points, $x_i = 1 + \dfrac{2i}{n}$.

$$\begin{aligned}
A_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \sum_{i=1}^{n} \left[x_i{}^2 + 3x_i\right] \frac{2}{n} \\
&= \frac{2}{n}\sum_{i=1}^{n} \left[\left(1+\frac{2i}{n}\right)^2 + 3\left(1+\frac{2i}{n}\right)\right] \\
&= \frac{2}{n}\sum_{i=1}^{n} \left(4 + \frac{10i}{n} + \frac{4i^2}{n^2}\right) \\
&= 8 + \frac{20}{n^2}\sum_{i=1}^{n} i + \frac{8}{n^3}\sum_{i=1}^{n} i^2 \\
&= 8 + \frac{20}{n^2}\left(\frac{n(n+1)}{2}\right) \\
&\quad + \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= 8 + \frac{10}{n}(n+1) + \frac{4}{3n^2}\left(2n^2+3n+1\right)
\end{aligned}$$

Now, to compute the exact area, we take the limit as $n \to \infty$:

$$\begin{aligned}
A &= \lim_{n\to\infty} A_n \\
&= \lim_{n\to\infty} \\
&\left[8 + \frac{10}{n}(n+1) + \frac{4}{3n^2}\left(2n^2+3n+1\right)\right] \\
&= 8 + 10 + \frac{8}{3} = \frac{62}{3}
\end{aligned}$$

13. (a) $\Delta x = \dfrac{1}{n}$. We will use right endpoints as evalution points, $x_i = \dfrac{i}{n}$.

$$\begin{aligned}
A_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \frac{1}{n}\sum_{i=1}^{n} \left[2\left(\frac{i}{n}\right)^2 + 1\right] \\
&= \frac{2}{n^3}\sum_{i=1}^{n} i^2 + 1 \\
&= \frac{2}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] + 1 \\
&= \frac{\left(5n^2+n+1\right)}{3n^2}
\end{aligned}$$

Now, to compute the exact area, we take the limit as $n \to \infty$:

$$\begin{aligned}
A &= \lim_{n\to\infty} A_n \\
&= \lim_{n\to\infty} \left[\frac{\left(5n^2+n+1\right)}{3n^2}\right] = \frac{5}{3}.
\end{aligned}$$

(b) $\Delta x = \dfrac{2}{n}$. We will use right endpoints as evalution points, $x_i = -1 + \dfrac{2i}{n}$.

$$\begin{aligned}
A_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \sum_{i=1}^{n} \left(2x_i{}^2 + 1\right)\left(\frac{2}{n}\right) \\
&= \frac{2}{n}\sum_{i=1}^{n} \left(2\left(-1+\frac{2i}{n}\right)^2 + 1\right) \\
&= \frac{2}{n}\sum_{i=1}^{n} \left(3 - \frac{8i}{n} + \frac{8i^2}{n^2}\right) \\
&= 6 - \frac{16}{n^2}\sum_{i=1}^{n} i + \frac{16}{n^3}\sum_{i=1}^{n} i^2 \\
&= 6 - \frac{16}{n^2}\left(\frac{n(n+1)}{2}\right) \\
&\quad + \frac{16}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= 6 - \left(\frac{8n+8}{n}\right) + \left(\frac{16n^2+24n+8}{3n^2}\right)
\end{aligned}$$

Now, to compute the exact area, we take the limit as $n \to \infty$:

$$\begin{aligned}
A &= \lim_{n\to\infty} A_n \\
&= \lim_{n\to\infty} \\
&\left[6 - \left(\frac{8n+8}{n}\right) + \left(\frac{16n^2+24n+8}{3n^2}\right)\right] \\
&= 6 - 8 + \frac{16}{3} = \frac{10}{3}
\end{aligned}$$

(c) $\Delta x = \dfrac{2}{n}$. We will use right endpoints as evaluation points, $x_i = 1 + \dfrac{2i}{n}$.

$$\begin{aligned}
A_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \frac{2}{n}\sum_{i=1}^{n} 2\left(1+\frac{2i}{n}\right)^2 + 1 \\
&= \frac{2}{n}\sum_{i=1}^{n} \left(\frac{8i^2}{n^2} + \frac{8i}{n} + 3\right) \\
&= \frac{16}{n^3}\sum_{i=1}^{n} i^2 + \frac{16}{n^2}\sum_{i=1}^{n} i + 6 \\
&= \frac{16}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&\quad + \frac{16}{n^2}\left(\frac{n(n+1)}{2}\right) + 6 \\
&= \frac{16n(n+1)(2n+1)}{6n^3}
\end{aligned}$$

$$+\frac{16n(n+1)}{2n^2}+6$$

Now to compute the exact area, we take the limit as $n \to \infty$:

$$A = \lim_{n\to\infty} A_n$$
$$= \lim_{n\to\infty}\left(\frac{16n(n+1)(2n+1)}{6n^3} + \frac{16n(n+1)}{2n^2}+6\right)$$
$$= \lim_{n\to\infty} \frac{32}{6}+\frac{16}{2}+6 \quad = \frac{58}{3}$$

14. (a) $\Delta x = \frac{1}{n}$. We will use right endpoints as evalution points, $x_i = \frac{i}{n}$.

$$A_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left(4{x_i}^2 - x_i\right)\frac{1}{n}$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left[4\left(\frac{i}{n}\right)^2 - \left(\frac{i}{n}\right)\right]$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{4i^2}{n^2}-\frac{i}{n}\right)\right]$$
$$= \frac{4}{n}\sum_{i=1}^{n}\frac{i^2}{n^2} - \frac{1}{n}\sum_{i=1}^{n}\frac{i}{n}$$
$$= \frac{4}{n^3}\left(\frac{n\,(n+1)\,(2n+1)}{6}\right) - \frac{1}{n^2}\left(\frac{n\,(n+1)}{2}\right)$$
$$= \frac{2}{3n^2}\left(2n^2+3n+1\right) - \frac{1}{2n}(n+1)$$
$$= \frac{5}{6}+\frac{3}{2n}+\frac{2}{3n^2}$$

Now, to compute the exact area, we take the limit as $n \to \infty$:

$$A = \lim_{n\to\infty} A_n$$
$$= \lim_{n\to\infty}\left(\frac{5}{6}+\frac{3}{2n}+\frac{2}{3n^2}\right)$$
$$= \frac{5}{6}$$

(b) $\Delta x = \frac{2}{n}$. We will use right endpoints as evalution points, $x_i = -1+\frac{2i}{n}$.

$$A_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left[4{x_i}^2 - x_i\right]\frac{2}{n}$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left[4\left(-1+\frac{2i}{n}\right)^2 - \left(-1+\frac{2i}{n}\right)\right]$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left(5-\frac{18i}{n}+\frac{16i^2}{n^2}\right)$$
$$= \frac{10}{n}\sum_{i=1}^{n}1 - \frac{36}{n^2}\sum_{i=1}^{n}i + \frac{32}{n^3}\sum_{i=1}^{n}i^2$$
$$= 10 - \frac{36}{n^2}\left(\frac{n\,(n+1)}{2}\right) + \frac{32}{n^3}\left(\frac{n\,(n+1)\,(2n+1)}{6}\right)$$
$$= 10 - \frac{18}{n}(n+1) + \frac{16}{3n^2}\left(2n^2+3n+1\right)$$
$$= \frac{8}{3}-\frac{2}{n}+\frac{16}{3n^2}$$

Now, to compute the exact area, we take the limit as $n \to \infty$:

$$A = \lim_{n\to\infty} A_n$$
$$= \lim_{n\to\infty}\left(\frac{8}{3}-\frac{2}{n}+\frac{16}{3n^2}\right)$$
$$= \frac{8}{3}$$

(c) $\Delta x = \frac{2}{n}$. We will use right endpoints as evaluation points $x_i = 1+\frac{2i}{n}$.

$$A_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left[4{x_i}^2 - x_i\right]\frac{2}{n}$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left[4\left(1+\frac{2i}{n}\right)^2 - \left(1+\frac{2i}{n}\right)\right]$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left(3+\frac{14i}{n}+\frac{16i^2}{n^2}\right)$$
$$= \frac{6}{n}\sum_{i=1}^{n}1 + \frac{28}{n^2}\sum_{i=1}^{n}i + \frac{32}{n^3}\sum_{i=1}^{n}i^2$$
$$= 6 + \frac{28}{n^2}\left(\frac{n\,(n+1)}{2}\right) + \frac{32}{n^3}\left(\frac{n\,(n+1)\,(2n+1)}{6}\right)$$
$$= 6 + \frac{14}{n}(n+1) + \frac{16}{3n^2}\left(2n^2+3n+1\right)$$
$$= \frac{92}{3}+\frac{30}{n}+\frac{16}{3n^2}$$

Now, to compute the exact area, we take the limit as $n \to \infty$:

$$A = \lim_{n\to\infty} A_n$$
$$= \lim_{n\to\infty}\left(\frac{92}{3}+\frac{30}{n}+\frac{16}{3n^2}\right)$$
$$= \frac{92}{3}$$

15.

n	Left Endpoint	Midpoint	Right Endpoint
10	10.56	10.56	10.56
50	10.662	10.669	10.662
100	10.6656	10.6672	10.6656
500	10.6666	10.6667	10.6666
1000	10.6667	10.6667	10.6667
5000	10.6667	10.6667	10.6667

16.

n	Left Endpoint	Midpoint	Right Endpoint
10	0.91940	1.00103	1.07648
50	0.98421	1.00004	1.01563
100	0.99213	1.00001	1.00783
500	0.99843	1.00000	1.00157
1000	0.99921	1.00000	1.00079
5000	0.99984	1.00000	1.00016

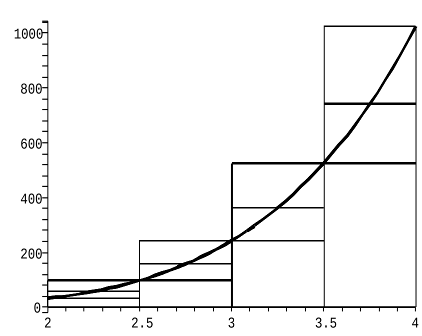
17.

n	Left Endpoint	Midpoint	Right Endpoint
10	15.48000	17.96000	20.68000
50	17.4832	17.9984	18.5232
100	17.7408	17.9996	18.2608
500	17.9480	17.9999	18.0520
1000	17.9740	17.9999	18.0260
5000	17.9948	17.9999	18.0052

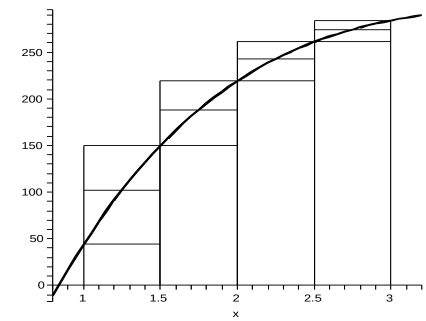
18.

n	Left Endpoint	Midpoint	Right Endpoint
10	-2.20000	-2	-1.80000
50	-2.04000	-2	-1.96000
100	-2.02000	-2	-1.98000
500	-2.00400	-2	-1.99600
1000	-2.00200	-2	-1.99800
5000	-2.00040	-2	-1.99960

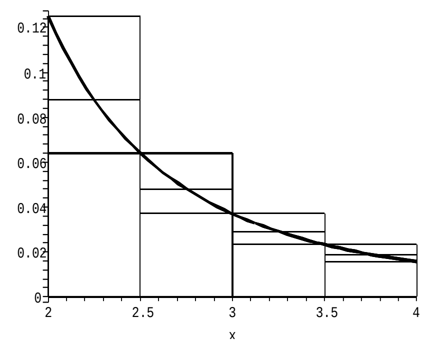
19. Let L, M, and R be the values of the Riemann sums with left endpoints, midpoints and right endpoints. Let A be the area under the curve. Then: $L < M < A < R$.



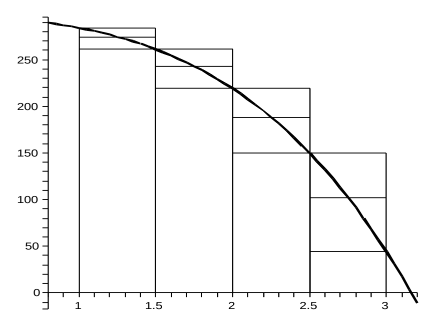
20. Let L, M, and R be the values of the Riemann sums with left endpoints, midpoints and right endpoints. Let A be the area under the curve. Then: $L < A < M < R$.



21. Let L, M, and R be the values of the Riemann sums with left endpoints, midpoints and right endpoints. Let A be the area under the curve. Then: $R < A < M < L$.



22. Let L, M, and R be the values of the Riemann sums with left endpoints, midpoints and right endpoints. Let A be the area under the curve. Then: $R < A < M < L$.



23. There are many possible answers here. One possibility is to use $x = 1/6$ on $[0, 0.5]$ and $x = \sqrt{23}/6$ on $[0.5, 1]$.

24. There are many possible answers here. One possibility is to use $x = 1/4$ on $[0, 0.5]$ and

$x = 25/36$ on $[0.5, 1]$.

25. (a) We subdivide the interval $[a, b]$ into n equal subintervals. If you are located at $a + (b-a)/n$ (the first right endpoint), then each step of distance Δx takes you to a new right endpoint. To arrive at the i-th right endpoint, you have to take $(i-1)$ steps to the right of distance Δx. Therefore,
$c_i = a + (b-a)/n + (i-1)\Delta x = a + i\Delta x$.

(b) We subdivide the interval $[a, b]$ into n equal subintervals. The first evaluation point is $a + \Delta x/2$. From this evaluation point, each step of distance Δx takes you to a new evaluation point. To arrive at the i-th evaluation point, you have to take $(i-1)$ steps to the right of distance Δx. Therefore,
$c_i = a + \Delta x/2 + (i-1)\Delta x$
$= a + (i - 1/2)\,\Delta x$, for $i = 1, \ldots, n$.

26. (a) We subdivide the interval $[a, b]$ into n equal subintervals. If you are located at a (the first left endpoint), then each step of distance Δx takes you to a new left endpoint. To arrive at the i-th left endpoint, you have to take $(i-1)$ steps to the right of distance Δx. Therefore,
$c_i = a + (i-1)\Delta x$.

(b) We subdivide the interval $[a, b]$ into n equal subintervals. The first evaluation point is $a + \Delta x/3$. From this evaluation point, each step of distance Δx takes you to a new evaluation point. To arrive at the i-th evaluation point, you have to take $(i-1)$ steps to the right of distance Δx. Therefore,
$c_i = a + \Delta x/3 + (i-1)\Delta x$
$= a + (i - 2/3)\,\Delta x$, for $i = 1, \ldots, n$.

27. Consider interval $[2, 4]$, then $\Delta x = \dfrac{2}{n}$.
Use right endpoints as evaluation points, $x_i = \left(2 + \dfrac{2i}{n}\right)$.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\sqrt{2 + \frac{2i}{n}}\right)\frac{2}{n}\right]$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\sqrt{2}\left(\sqrt{1 + \frac{i}{n}}\right)\frac{2}{n}\right]$$

Hence,
$$A_2 = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\sqrt{2}\left(\sqrt{1 + \frac{i}{n}}\right)\frac{2}{n}\right].$$

28. Consider interval$[0, 2]$, then $\Delta x = \dfrac{2}{n}$.
Use mid points as evaluation points, $x_i = \dfrac{\left(\frac{2(i-1)}{n} + \frac{2i}{n}\right)}{2}$.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\sqrt{\frac{\frac{2(i-1)}{n} + \frac{2i}{n}}{2}}\right)\frac{2}{n}\right]$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\sqrt{\frac{2i - 2 + 2i}{2n}}\right)\frac{2}{n}\right]$$

Hence,
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{1}{\sqrt{n}}\left(\sqrt{2i-1}\right)\frac{2}{n}\right].$$

Assume
$i = k + 1$.
$$A = \sum_{k=0}^{n-1} \left[\frac{1}{\sqrt{n}}\left(\sqrt{2\,(k+1) - 1}\right)\frac{2}{n}\right]$$
$$= \sum_{k=1}^{n} \left[\frac{1}{\sqrt{n}}\left(\sqrt{2k+1}\right)\frac{2}{n}\right]$$

hence,
$$A_1 = \sum_{k=1}^{n} \left[\frac{1}{\sqrt{n}}\left(\sqrt{2k+1}\right)\frac{2}{n}\right].$$

29. $U_4 = \dfrac{2}{4}\displaystyle\sum_{i=1}^{4}\left(\frac{i}{2}\right)^2$

$$= \frac{1}{8}\sum_{i=1}^{4} i^2 = \frac{1}{8}\left[1^2 + 2^2 + 3^2 + 4^2\right]$$
$$= \frac{30}{8} = 3.75 \; L_4 = \frac{2}{4}\sum_{i=1}^{4}\left(\frac{i-1}{2}\right)^2$$
$$= \frac{1}{8}\sum_{i=1}^{4} i^2 = \frac{1}{8}\left[0^2 + 1^2 + 2^2 + 3^2\right]$$
$$= \frac{14}{8} = 1.75$$

30. The function $f(x) = x^2$ is symmetric on the two intervals $[-2, 0]$ and $[0, 2]$, so the upper sum U_8 is just double the value of U_4 as calculated in Exercise 35, and the same is for L_8. The answers are
$U_8 = 2 \cdot 3.75 = 7.5, L_8 = 2 \cdot 1.75 = 3.5$.

31. (a) $U_n = \dfrac{2}{n}\displaystyle\sum_{i=1}^{n}\left(\frac{2i}{n}\right)^2$

$$= \left(\frac{2}{n}\right)^3 \sum_{i=1}^{n} i^2$$
$$= \left(\frac{2}{n}\right)^3 \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{4}{3}\frac{n(n+1)(2n+1)}{n^3}$$
$$= \frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$
$$\lim_{n\to\infty} U_n = \frac{4}{3}(2) = \frac{8}{3}$$

(b) $L_n = \frac{2}{n}\sum_{i=1}^{n}\left(\frac{2(i-1)}{n}\right)^2$
$$= \left(\frac{2}{n}\right)^3 \sum_{i=1}^{n}(i-1)^2$$
$$= \left(\frac{2}{n}\right)^3 \sum_{i=1}^{n-1} i^2$$
$$= \left(\frac{2}{n}\right)^3 \frac{(n-1)(n)(2n-1)}{6}$$
$$= \frac{4}{3}\frac{(n-1)(n)(2n-1)}{n^3}$$
$$= \frac{4}{3}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$$
$$\lim_{n\to\infty} L_n = \frac{4}{3}(2) = \frac{8}{3}$$

32. (a) $U_n = \frac{2}{n}\sum_{i=1}^{n}\left[\left(0+\frac{2}{n}i\right)^3+1\right]$
$$= \frac{2}{n}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^3+1\right]$$
$$= \left(\frac{2}{n}\right)^4 \sum_{i=1}^{n} i^3 + \sum_{i=1}^{n} 1$$
$$= \frac{2^4}{n^4}\left[\frac{n^2(n+1)^2}{4}+\frac{2}{n}(n)\right]$$
$$= \frac{4(n+1)^2}{n^2}+2$$
$$= \frac{4(n^2+2n+1)}{n^2}+2$$
$$= 6+\frac{8}{n}+\frac{4}{n^2}$$
$$\lim_{n\to\infty} U_n = 6$$

(b) $L_n = \frac{2}{n}\sum_{i=0}^{n-1}\left[\left(0+\frac{2}{n}i\right)^3+1\right]$
$$= \frac{2}{n}\sum_{i=0}^{n-1}\left[\left(\frac{2i}{n}\right)^3+1\right]$$
$$= \left(\frac{2}{n}\right)^4 \sum_{i=0}^{n-1} i^3 + \sum_{i=1}^{n} 1$$
$$= \frac{2^4}{n^4}\left[\frac{(n-1)^2n^2}{4}+\frac{2}{n}(n)\right]$$
$$= \frac{4(n-1)^2}{n^2}+2$$
$$= \frac{4(n^2-2n+1)}{n^2}+2$$
$$= 6-\frac{8}{n}+\frac{4}{n^2}$$
$$\lim_{n\to\infty} L_n = 6$$

33. Here, $f(x) = a^2 - x^2$ and interval is $[-a,\, a]$.
Hence $\Delta x = \frac{2a}{n}$.
Use right endpoints as evaluation points, $x_i = \left(-a+\frac{2ai}{n}\right)$.
$$A_n = \sum_{i=1}^{n} f(x_i)\Delta x$$
$$= \sum_{i=1}^{n}\left(a^2 - {x_i}^2\right)\Delta x$$
$$= \sum_{i=1}^{n}\left[\left(a^2-\left(-a+\frac{2ia}{n}\right)^2\right)\frac{2a}{n}\right]$$
$$= \sum_{i=1}^{n}\left[\left(\frac{4ia^2}{n}-\frac{4i^2a^2}{n^2}\right)\frac{2a}{n}\right]$$
$$= \frac{8a^3}{n^2}\sum_{i=1}^{n} i - \frac{8a^3}{n^3}\sum_{i=1}^{n} i^2$$
$$= \frac{8a^3}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{8a^3}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$
$$= \frac{4a^3}{n}(n+1) - \frac{4a^3}{3n^3}\left(2n^2+3n+1\right)$$
Now, to compute the exact area, we take the limit as $n\to\infty$:
$$A = \lim_{n\to\infty} A_n$$
$$= \lim_{n\to\infty}\left[\frac{4a^3}{n}(n+1)-\frac{4a^3}{3n^3}\left(2n^2+3n+1\right)\right]$$
$$= \left(4-\frac{8}{3}\right)a^3 = \frac{4}{3}a^3$$
$$= \frac{2}{3}(2a)\left(a^2\right)$$

34. Here, $f(x) = ax^2$ and interval is $[0,\, b]$.
Hence $\Delta x = \frac{b}{n}$.
Use right endpoints as evaluation points, $x_i = \left(\frac{bi}{n}\right)$.
$$A_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \sum_{i=1}^{n} \left(a{x_i}^2\right) \Delta x$$
$$= \sum_{i=1}^{n} \left[a\left(\frac{bi}{n}\right)^2 \frac{b}{n}\right]$$
$$= \frac{ab^3}{n^3} \sum_{i=1}^{n} i^2$$
$$= \frac{ab^3}{n^3} \left(\frac{n(n+1)(2n+1)}{6}\right)$$
$$= \frac{ab^3}{6n^2} \left(2n^2 + 3n + 1\right)$$

Now, to compute the exact area, we take the limit as $n \to \infty$:

$$A = \lim_{n\to\infty} A_n$$
$$= \lim_{n\to\infty} \left[\frac{ab^3}{6n^2} \left(2n^2 + 3n + 1\right)\right]$$
$$= \frac{2ab^3}{6} = \frac{ab^3}{3} = \frac{1}{3} b \left(ab^2\right)$$

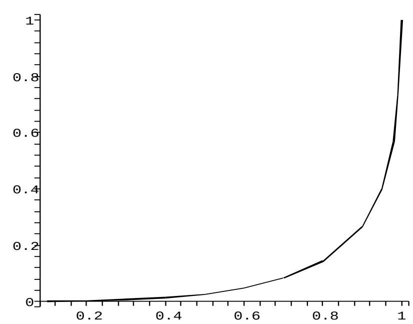
35. Using left hand endpoints:
$L_8 = [f(0.0)+f(0.1)+f(0.2)+f(0.3)+f(0.4)+f(0.5)+f(0.6)+f(0.7)](0.1)$
$= (2.0 + 2.4 + 2.6 + 2.7 + 2.6 + 2.4 + 2.0 + 1.4)(0.1) = 1.81$
Right endpoints:
$R_8 = [f(0.1)+f(0.2)+f(0.3)+f(0.4)+f(0.5)+f(0.6)+f(0.7)+f(0.8)](0.2)$
$= (2.4 + 2.6 + 2.7 + 2.6 + 2.4 + 2.0 + 1.4 + 0.6)(0.1) = 1.67$

36. Using left hand endpoints:
$L_8 = [f().0)+f(0.2)+f(0.4)+f(0.6)+f(0.8)+f(1.0)+f(1.2)+f(1.4)](0.2)$
$= (2.0 + 2.2 + 1.6 + 1.4 + 1.6 + 2.0 + 2.2 + 2.4)(0.2) = 3.08$
Right endpoints:
$R_8 = [f(0.2)+f(0.4)+f(0.6)+f(0.8)+f(1.0)+f(1.2)+f(1.4)+f(1.6)](0.2)$
$= (2.2 + 1.6 + 1.4 + 1.6 + 2.0 + 2.2 + 2.4 + 2.0)(0.2) = 3.08$

37. Using left hand endpoints:
$L_8 = [f(1.0)+f(1.1)+f(1.2)+f(1.3)+f(1.4)+f(1.5)+f(1.6)+f(1.7)](0.1)$
$= (1.8 + 1.4 + 1.1 + 0.7 + 1.2 + 1.4 + 1.82 + 2.4)(0.1) = 1.182$
Right endpoints:
$R_8 = [f(1.1)+f(1.2)+f(1.3)+f(1.4)+f(1.5)+f(1.6)+f(1.7)+f(1.8)](0.1)$
$= (1.4 + 1.1 + 0.7 + 1.2 + 1.4 + 1.82 + 2.4 + 2.6)(0.1) = 1.262$

38. Using left hand endpoints:
$L_8 = [f(1.0)+f(1.2)+f(1.4)+f(1.6)+f(1.8)+f(2.0)+f(2.2)+f(2.4)](0.2)$
$= (0.0 + 0.4 + 0.6 + 0.8 + 1.2 + 1.4 + 1.2 + 1.4)(0.2) = 1.40$
Right endpoints:
$R_8 = [f(1.2)+f(1.4)+f(1.6)+f(1.8)+f(2.0)+f(2.2)+f(2.4)+f(2.6)](0.2)$
$= (0.4 + 0.6 + 0.8 + 1.2 + 1.4 + 1.2 + 1.4 + 1.0)(0.2) = 1.60$

39. $A \approx (0.2-0.1)(0.002) + (0.3-0.2)(0.004) + (0.4-0.3)(0.008) + (0.5-0.4)(0.014) + (0.6-0.5)(0.026) + (0.7-0.6)(0.048) + (0.8-0.7)(0.085) + (0.9-0.8)(0.144) + (0.95-0.9)(0.265) + (0.98-0.95)(0.398) + (0.99-0.98)(0.568) + (1-0.99)(0.736) + 1/2 \cdot [(0.1-0)(0.002)$
$+(0.2-0.1)(0.004-0.002)+(0.3-0.2)(0.008-0.004) + (0.4-0.3)(0.014-0.008) + (0.5-0.4)(0.026-0.014) + (0.6-0.5)(0.048-0.026) + (0.7-0.6)(0.085-0.048) + (0.8-0.7)(0.144-0.085) + (0.9-0.8)(0.265-0.144) + (0.95-0.9)(0.398-0.265) + (0.98-0.95)(0.568-0.398) + (0.99-0.98)(0.736-0.568)\ (1-0.99)(1-0.736)]$
≈ 0.092615 The Lorentz curve looks like:



40. Obviously $G = A_1/A_2$ is greater or equal to 0. From the above figure we see that the Lorentz curve is below the diagonal line $y = x$ on the interval $[0, 1]$, hence the area $A_1 \leq$ the area A_2. Furthermore, $A_2 =$ the area of the triangle formed by the points $(0, 0), (1, 0)$ and $(1, 1)$, hence equal to $1/2$. Now $G = A_1/A_2 = 2A_1$. Using the date in Exercise 33, $G \approx 2 \cdot 0.092615 = 0.185230$.

4.4 The Definite Integral

1. We know that
$$\int_0^3 \left(x^3 + x\right) dx \approx \sum_{i=1}^{n} \left(c_i^3 + c_i\right) \Delta x$$
Where $c_i = \dfrac{x_i + x_{i-1}}{2}, x_i = \dfrac{3i}{n}, n = 6.$

Here $c_i = \dfrac{\frac{3i}{6} + \frac{3(i-1)}{6}}{2} = \dfrac{(2i-1)}{4}$.

$\sum\limits_{i=1}^{n} \left(c_i^3 + c_i\right) . \dfrac{3}{n}$

$= \sum\limits_{i=1}^{6} \left[\dfrac{(2i-1)^3}{64} + \dfrac{(2i-1)}{4}\right] . \dfrac{1}{2}$

$= \Bigg(\dfrac{1}{64} + \dfrac{1}{4} + \dfrac{27}{64} + \dfrac{3}{4} + \dfrac{125}{64} + \dfrac{5}{4} + \dfrac{343}{64}$

$+ \dfrac{7}{4} + \dfrac{729}{64} + \dfrac{9}{4} + \dfrac{1331}{64} + \dfrac{11}{4}\Bigg) . \dfrac{1}{2}$

$\Rightarrow \int_0^3 \left(x^3 + x\right) dx \approx 24.47$

2. We know that

$\int_0^3 \sqrt{x^2+1}dx \approx \sum\limits_{i=1}^{n} \sqrt{c_i^2 + 1}\Delta x$

Where $c_i = \dfrac{x_i + x_{i-1}}{2}, x_i = \dfrac{3i}{n}, n = 6$.

Here $c_i = \dfrac{\frac{3i}{6} + \frac{3(i-1)}{6}}{2} = \dfrac{(2i-1)}{4}$.

$\sum\limits_{i=1}^{n} \sqrt{c_i^2 + 1}\left(\dfrac{3}{n}\right)$

$= \sum\limits_{i=1}^{6} \left(\sqrt{\left(\dfrac{2i-1}{4}\right)^2 + 1}\right) . \dfrac{1}{2}$

$= \Bigg(\dfrac{\sqrt{17}}{4} + \dfrac{5}{4} + \dfrac{\sqrt{41}}{4} + \dfrac{\sqrt{65}}{4}$

$+ \dfrac{\sqrt{97}}{4} + \dfrac{\sqrt{137}}{4}\Bigg) . \dfrac{1}{2}$

$\Rightarrow \int_0^3 \sqrt{x^2+1}dx \approx 5.64$

3. We know that

$\int_0^\pi \sin x^2 dx \approx \sum\limits_{i=1}^{n} \left(\sin c_i^2\right) \Delta x$.

Where $c_i = \dfrac{x_i + x_{i-1}}{2}, x_i = \dfrac{i\pi}{n}, n = 6$.

Here$c_i = \dfrac{\frac{\pi i}{6} + \frac{\pi(i-1)}{6}}{2} = \dfrac{(2i-1)\pi}{12}$.

$\sum\limits_{i=1}^{n} \left(\sin c_i^2\right)\left(\dfrac{\pi}{n}\right)$

$= \sum\limits_{i=1}^{6} \left[\sin\left(\dfrac{(2i-1)\pi}{12}\right)^2\right] . \left(\dfrac{\pi}{6}\right)$

$= \Bigg[\sin\left(\dfrac{\pi}{12}\right)^2 + \sin\left(\dfrac{3\pi}{12}\right)^2 + \sin\left(\dfrac{5\pi}{12}\right)^2$

$+ \sin\left(\dfrac{7\pi}{12}\right)^2 + \sin\left(\dfrac{9\pi}{12}\right)^2 + \sin\left(\dfrac{11\pi}{12}\right)^2\Bigg] . \dfrac{\pi}{6}$

$\Rightarrow \int_0^\pi \sin x^2 dx \approx 0.8685$

4. We know that

$\int_{-2}^{2} e^{-x^2} dx \approx \sum\limits_{i=1}^{n} e^{-c_i^2} \Delta x$.

Where $c_i = \dfrac{x_i + x_{i-1}}{2}, x_i = -2 + \dfrac{4i}{n}, n = 6$.

Here,

$c_i = \dfrac{\left(-2 + \frac{4i}{6}\right) + \left[-2 + \frac{4(i-1)}{6}\right]}{2} = \dfrac{2i-7}{3}$.

$\sum\limits_{i=1}^{n} e^{-c_i^2}\left(\dfrac{4}{n}\right) = \sum\limits_{i=1}^{6} e^{-c_i^2}\left(\dfrac{4}{6}\right)$

$= \Big[e^{-25/9} + e^{-1} + e^{-1/9}$

$+ e^{-1/9} + e^{-1} + e^{-25/9}\Big] . \dfrac{2}{3}$

$= \left[e^{-25/9} + e^{-1} + e^{-1/9}\right] . \dfrac{4}{3}$

$\Rightarrow \int_{-2}^{2} e^{-x^2} dx \approx 1.7665$

5. Notice that the graph of $y = x^2$ is above the x-axis. So,$\int_1^3 x^2 dx$ is the area of the region bounded by $y = x^2$ and the x-axis, between $x = 1$ and $x = 3$.

6. Notice that the graph of $y = e^x$ is above the x-axis. So, $\int_0^1 e^x dx$ is the area of the region bounded by $y = e^x$, and the x-axis, between $x = 0$ and $x = 1$.

7. Notice that the graph of $y = x^2 - 2$ is below the $x-$axis for $|x| \le \sqrt{2}$ above the, $x-$axis for $|x| \ge \sqrt{2}$.

Also,

$\int_0^2 \left(x^2 - 2\right) dx$

$= \int_0^{\sqrt{2}} \left(x^2 - 2\right) dx + \int_{\sqrt{2}}^2 \left(x^2 - 2\right) dx$.

So, $\int_0^2 \left(x^2 - 2\right) dx$ is the additon of the areas of the regions bounded by $y = x^2 - 2$and the $x-$axis, between $x = 0$ and $x = \sqrt{2}$ (which is below the $x-$axis) and between $x = \sqrt{2}$ and $x = 2$ (which is above the $x-$axis)

8. Notice that the graph of $y = x^3 - 3x^2 + 2x$ is below the x-axis, for $1 \le x \le 2$ and $x \le 0$ and above the x-axis, for all other values of x.

Also,

$\int_0^2 \left(x^3 - 3x^2 + 2x\right) dx$

$= \int_0^1 \left(x^3 - 3x^2 + 2x\right) dx$

$+ \int_1^2 \left(x^3 - 3x^2 + 2x\right) dx$

So, $\int_0^2 \left(x^3 - 3x^2 + 2x\right) dx$ is the additon of the areas of the regions bounded by $y = x^3 - 3x^2 + 2x$ and the x-axis between $x = 0$ and $x = 1$ (which is above the x-axis) and between $x = 1$ and $x = 2$ (which is below the x-axis).

9. For n rectangles, $\Delta x = \frac{1}{n}$, $x_i = i\Delta x$.

$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$

$= \sum_{i=1}^{n} 2x_i \; \Delta x = \frac{1}{n}\sum_{i=1}^{n} 2\left(\frac{i}{n}\right) = \frac{2}{n^2}\sum_{i=1}^{n} i$

$= \frac{2}{n^2}\left(\frac{n(n+1)}{2}\right) = \frac{(n+1)}{n}$

To compute the value of the integral, we take the limit as $n \to \infty$,

$\int_0^1 2x dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)}{n} = 1$

10. For n rectangles, $\Delta x = \frac{1}{n}$, $x_i = 1 + i\Delta x$.

$R_n = \sum_{i=1}^{n} f(x_i) \; \Delta x$

$= \sum_{i=1}^{n} 2x_i \; \Delta x = \frac{1}{n}\sum_{i=1}^{n} 2\left(1 + \frac{i}{n}\right)$

$= \frac{2}{n}\sum_{i=1}^{n} 1 + \frac{2}{n^2}\sum_{i=1}^{n} i$

$= \frac{2}{n}(n) + \frac{2}{n^2}\left(\frac{n(n+1)}{2}\right)$

$= 2 + \frac{(n+1)}{n}$

To compute the value of the integral, we take the limit as $n \to \infty$,

$\int_1^2 2x dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} 2 + \frac{(n+1)}{n}$

$= 2 + 1 = 3$

11. For n rectangles,
$\Delta x = \frac{2}{n}, x_i = i\Delta x = \frac{2i}{n}$.

$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$

$= \sum_{i=1}^{n}(x_i^2)\Delta x = \frac{2}{n}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)^2$

$= \frac{2}{n}\sum_{i=1}^{n}\frac{4i^2}{n^2} = \frac{8}{n^3}\sum_{i=1}^{n} i^2$

$= \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$

$= \frac{4(n+1)(2n+1)}{3n^2}$

To compute the value of the integral, we take the limit as $n \to \infty$,

$\int_0^2 x^2 dx = \lim_{n\to\infty} R_n$

$= \lim_{n\to\infty} \frac{4(n+1)(2n+1)}{3n^2} = \frac{8}{3}$

12. For n rectangles,
$\Delta x = \frac{3}{n}, x_i = i\Delta x = \frac{3i}{n}$.

$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$

$= \sum_{i=1}^{n}(x_i^2 + 1)\Delta x = \frac{3}{n}\sum_{i=1}^{n} 2\left(\frac{3i}{n}\right)^2 + 1$

$= \frac{3}{n}\sum_{i=1}^{n}\frac{18i^2}{n^2} + 1$

$= \frac{54}{n^3}\sum_{i=1}^{n} i^2 + \frac{3}{n}\sum_{i=1}^{n} 1$

$= \frac{54}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \left(\frac{3}{n}\right)n$

$= \frac{9(n+1)(2n+1)}{n^2} + 3$

To compute the value of the integral, we take the limit as $n \to \infty$,

$\int_0^3 (x^2+1)dx = \lim_{n\to\infty} R_n$

$= \lim_{n\to\infty} \frac{9(n+1)(2n+1)}{n^2} + 3$

$= 9 + 3 = 12$

13. For n rectangles, $\Delta x = \frac{2}{n}$,
$x_i = 1 + i\Delta x = 1 + \frac{2i}{n}$

$R_n = \sum_{i=1}^{n} f(x_i) \; \Delta x$

$= \sum_{i=1}^{n}(x_i^2 - 3)\Delta x$

$= \frac{2}{n}\sum_{i=1}^{n}\left[\left(1 + \frac{2i}{n}\right)^2 - 3\right]$

$= \sum_{i=1}^{n}\left(\frac{8i}{n^2} + \frac{8i^2}{n^3} - \frac{4}{n}\right)$

$= \frac{8n(n+1)}{2n^2} + \frac{8n(n+1)(2n+1)}{6n^3} - 4$

To compute the value of the integral, we take the limit as $n \to \infty$,

$$\int_1^3 (x^2-3)dx = \lim_{n\to\infty} R_n$$

$$= \frac{8}{2} + \frac{16}{6} - 4 = \frac{8}{3}$$

14. For n rectangles,

$$\Delta x = \frac{4}{n}, x_i = -2 + i\Delta x = -2 + \frac{4i}{n}$$

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} (x_i^2 - 1)\Delta x$$

$$= \frac{4}{n}\sum_{i=1}^{n}\left(-2+\frac{4i}{n}\right)^2 - 1$$

$$= \frac{4}{n}\sum_{i=1}^{n}\left(3 - \frac{16i}{n} + \frac{16i^2}{n^2}\right)$$

$$= \frac{12}{n}\sum_{i=1}^{n} 1 - \frac{64}{n^2}\sum_{i=1}^{n} i + \frac{64}{n^3}\sum_{i=1}^{n} i^2$$

$$= \left(\frac{12}{n}\right) n - \frac{64}{n^2}\left(\frac{n(n+1)}{2}\right)$$

$$+ \frac{64}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= 12 - \frac{32(n+1)}{n} + \frac{32(n+1)(2n+1)}{3n^2}$$

To compute the value of the integral, we take the limit as $n \to \infty$,

$$\int_{-2}^{2} (x^2-1)dx = \lim_{n\to\infty} R_n$$

$$= \lim_{n\to\infty}\left[12 - \frac{32(n+1)}{n} + \frac{32(n+1)(2n+1)}{3n^2}\right]$$

$$= 12 - 32 + \frac{64}{3} = \frac{4}{3}$$

15. Notice that the graph of $y = 4 - x^2$ is above the x-axis between $x = -2$ and $x = 2$:

$$\int_{-2}^{2} (4-x^2)dx$$

16. Notice that the graph of $y = 4x - x^2$ is above the x-axis between $x = 0$ and $x = 4$:

$$\int_0^4 (4x - x^2)dx$$

17. Notice that the graph of $y = x^2 - 4$ is below the x-axis between $x = -2$ and $x = 2$. Since we are asked for area and the area in question is below the x-axis, we have to be a bit careful.

$$\int_{-2}^{2} -(x^2-4)dx$$

18. Notice that the graph of $y = x^2 - 4x$ is below the x-axis between $x = 0$ and $x = 4$. Since we are asked for area and the area in question is below the x-axis, we have to be a bit careful.

$$\int_0^4 -(x^2-4x)dx$$

19. $\displaystyle\int_0^{\pi} \sin x dx$

20. $\displaystyle -\int_{-\pi/2}^{0} \sin x dx + \int_0^{\pi/4} \sin x dx$

21. The total distance is the total area under the curve whereas the total displacement is the signed area under the curve. In this case, from $t = 0$ to $t = 4$, the function is always positive so the total distance is equal to the total displacement. This means we want to compute the definite integral $\displaystyle\int_0^4 40(1-e^{-2t})dt$. We compute various right hand sums for different values of n:

n	R_n
10	146.9489200
20	143.7394984
50	141.5635684
100	140.7957790
500	140.1662293
1000	140.0865751

It looks like these are converging to about 140. So, the total distance traveled is approximately 140 and the final position is $s(b) \approx s(0) + 140 = 0 + 140 = 140$.

22. The total distance is the total area under the curve whereas the total displacement is the signed area under the curve. In this case, from $t = 0$ to $t = 4$, the function is always positive so the total distance is equal to the total displacement. This means we want to compute the definite integral $\int_0^4 30e^{-t/4}\ dt$. We compute various right hand sums for different values of n:

n	R_n
10	72.12494524
20	73.97390774
50	75.09845086
100	75.47582684
500	75.77863788
1000	75.81654616

It looks like these are converging to about 75.8. So, the total distance traveled is approximately 75.8 and the final position is
$s(b) \approx s(0) + 75.8 = -1 + 75.8 = 74.8$.

23. $\int_0^4 f(x)dx$
$= \int_0^1 f(x)dx + \int_1^4 f(x)dx$
$= \int_0^1 2xdx + \int_1^4 4dx$
$\int_0^1 2xdx$ is the area of a triangle with base 1 and height 2 and therefore has $area = \frac{1}{2}(1)(2) = 1$.
$\int_1^4 4dx$ is the area of a rectangle with base 3 and height 4 and therefore has $area = (3)(4) = 12$.
Therefore
$\int_0^4 f(x)dx = 1 + 12 = 13$

24. $\int_0^4 f(x)dx$
$= \int_0^2 f(x)dx + \int_2^4 f(x)dx$
$= \int_0^2 2dx + \int_2^4 3xdx$
$\int_0^2 2dx$ is the area of a square with base 2 and height 2 (it is, after all, a square) and therefore has $area = 4$.
$\int_2^4 3xdx$ is a trapezoid with height 3 and bases 6 and 12 and therefore has area (using the formula in the front of the text)
$area = \frac{1}{2}(6+12)(2) = 18$.
Therefore
$\int_0^4 f(x)dx = 4 + 18 = 22$

25. $f_{ave} = \frac{1}{4}\int_0^4 (2x+1)dx$
$= \frac{1}{4}\lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n}\left(\frac{8i}{n} + 1\right)$
$= \lim_{n\to\infty} \left(\frac{8n(n+1)}{2n^2} + 1\right)$
$= 4 + 1 = 5$

26. $f_{ave} = \frac{1}{1}\int_0^1 (x^2 + 2x)dx$
$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i^2}{n^2} + \frac{2i}{n}\right)$
$= \lim_{n\to\infty} \left(\frac{n(n+1)(2n+1)}{6n^3} + \frac{2n(n+1)}{n^2}\right)$
$= \frac{2}{6} + 2 = \frac{7}{3}$

27. $f_{ave} = \frac{1}{3-1}\int_1^3 (x^2 - 1)dx$
$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[\left(1 + \frac{2i}{n}\right)^2 - 1\right]$
$= \lim_{n\to\infty} \frac{1}{n}\left(\frac{4i}{n} + \frac{4i^2}{n^2}\right)$
$= \lim_{n\to\infty} \left(\frac{4n(n+1)}{2n^2} + \frac{4n(n+1)(2n+1)}{6n^3}\right)$
$= 2 + \frac{4}{3} = \frac{10}{3}$

28. $f_{ave} = \frac{1}{1-0}\int_0^1 (2x - 2x^2)dx$
$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left[2\left(\frac{i}{n}\right) - 2\left(\frac{i}{n}\right)^2\right]$
$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\frac{2i}{n} + \frac{2i^2}{n^2}\right)$
$= \lim_{n\to\infty} \left(\frac{2n(n+1)}{2n^2} + \frac{2n(n+1)(2n+1)}{6n^3}\right)$
$= 1 + \frac{2}{3} = \frac{5}{3}$

29. The function $f(x) = 3\cos x^2$ is decreasing on $[\pi/3, \pi/2]$. Therefore, on this interval, the maximum occurs at the left endpoint and is $f(\pi/3) = 3\cos(\pi^2/9)$. The minimum occurs at the right endpoint and is $f(\pi/2) = 3\cos(\pi^2/4)$. Using these to estimate the value of the integral gives the following inequality:
$$\frac{\pi}{6}\cdot(3\cos\frac{\pi^2}{4}) \le \int_{\pi/3}^{\pi/2} 3\cos x^2 dx \le \frac{\pi}{6}\cdot(3\cos\frac{\pi^2}{9})$$
$$-1.23 \le \int_{\pi/3}^{\pi/2} 3\cos x^2 dx \le 0.72$$

30. The function $f(x) = e^{-x^2}$ is decreasing on $[0, 1/2]$. Therefore, on this interval, the maximum occurs at the left endpoint and is $f(0) = 1$. The minimum occurs at the right endpoint and is $f(1/2) = e^{-1/4}$. Using these to estimate

the value of the integral gives the following inequality:
$\frac{1}{2}(e^{-1/4}) \le \int_0^{1/2} e^{-x^2}dx \le \frac{1}{2}(1)$
$0.3894 \le \int_0^{1/2} e^{-x^2}dx \le 0.5$

31. The function $f(x) = \sqrt{2x^2+1}$ is increasing on $[0,2]$. Therefore, on this interval, the maximum occurs at the right endpoint and is $f(2) = 3$. The minimum occurs at the left endpoint and is $f(0) = 1$. Using these to estimate the value of the integral gives the following inequality:
$(2)(1) \le \int_0^2 \sqrt{2x^2+1}dx \le (2)(3)$
$2 \le \int_0^2 \sqrt{2x^2+1}dx \le 6$

32. The function $f(x) = \frac{3}{x^3+2}$ is decreasing on $[-1,1]$. Therefore, on this interval, the maximum occurs at the left endpoint and is $f(-1) = 3$. The minimum occurs at the right endpoint and is $f(1) = 1$. Using these to estimate the value of the integral gives the following inequality:
$(2)(1) \le \int_{-1}^1 \frac{3}{x^3+2}dx \le (2)(3)$
$2 \le \int_{-1}^1 \frac{3}{x^3+2}dx \le 6$

33. We are looking for a value c, such that
$f(c) = \frac{1}{2-0}\int_0^2 3x^2dx$
Since $\int_0^2 3x^2dx = 8$, we want to find c so that $f(c) = 4$ or, $3c^2 = 4$
Solving this equation using the quadratic formula gives $c = \pm\frac{2}{\sqrt{3}}$
We are interested in the value that is in the interval $[0,2]$, so $c = \frac{2}{\sqrt{3}}$.

34. We are looking for a value c, such that
$f(c) = \frac{1}{1-(-1)}\int_{-1}^1 (x^2-2x)dx$
Since $\int_{-1}^1 (x^2-2x)dx = \frac{2}{3}$, we want to find c so that $f(c) = \frac{1}{3}$ or, $c^2 - 2c = \frac{1}{3}$

Solving this equation using the quadratic formula gives $c = \frac{3 \pm 2\sqrt{3}}{3}$
We are interested in the value that is in the interval $[-1,1]$, so $c = \frac{3-2\sqrt{3}}{3}$.

35. (a) $\int_0^2 f(x)dx + \int_2^3 f(x)dx = \int_0^3 f(x)dx$

(b) $\int_0^3 f(x)dx - \int_2^3 f(x)dx = \int_0^2 f(x)dx$

36. (a) $\int_0^2 f(x)dx + \int_2^1 f(x)dx = \int_0^1 f(x)dx$

(b) $\int_{-1}^2 f(x)dx + \int_2^3 f(x)dx = \int_{-1}^3 f(x)dx$

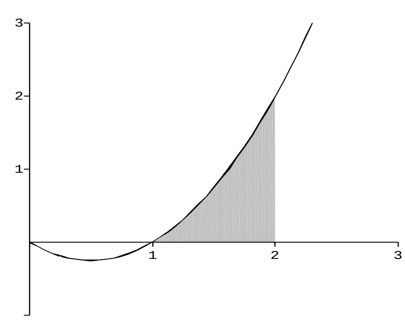
37. (a) $\int_1^3 (f(x)+g(x))\,dx$
$= \int_1^3 f(x)\,dx + \int_1^3 g(x)\,dx$
$= 3 + (-2) = 1$

(b) $\int_1^3 (2f(x)-g(x))\,dx$
$= 2\int_1^3 f(x)\,dx - \int_1^3 g(x)\,dx$
$= 2(3) - (-2) = 8$

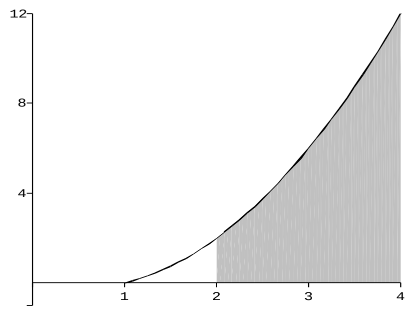
38. (a) $\int_1^3 (f(x)-g(x))\,dx$
$= \int_1^3 f(x)\,dx - \int_1^3 g(x)\,dx$
$= 3 - (-2) = 5$

(b) $\int_1^3 (4g(x)-3f(x))\,dx$
$= 4\int_1^3 g(x)\,dx - 3\int_1^3 f(x)\,dx$
$= 4(-2) - 3(3) = -17$

39. (a)

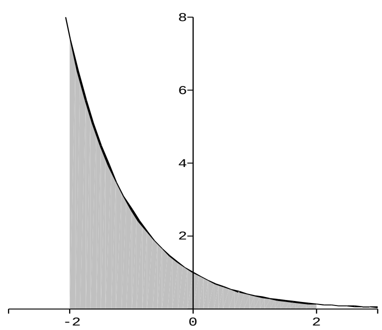


(b)



40. (a)

(b)



41. (a) Notice that $x^2 \sin x$ is a continuous function for all values of x and
for $1 \le x \le 2$,
$\sin x \le x^2 \sin x \le 4 \sin x$.
On using theorem 4.3,we get

$$\int_1^2 \sin x dx \le \int_1^2 x^2 \sin x dx \le \int_1^2 4 \sin x dx$$

$$(\cos 1 - \cos 2) \le \int_1^2 x^2 \sin x dx \le 4(\cos 1 - \cos 2)$$

(b) Notice that $x^2 \sin x$ is a continuous function for all values of x and
for $1 \le x \le 2, x^2 \sin 1 \le x^2 \sin x \le x^2$.
On using theorem 4.3,we get

$$\sin 1 \int_1^2 x^2 dx \le \int_1^2 x^2 \sin x dx \le \int_1^2 x^2 dx$$

$$\sin 1 \left. \frac{x^3}{3} \right|_1^2 \le \int_1^2 x^2 \sin x dx \le \left. \frac{x^3}{3} \right|_1^2$$

$$\frac{7}{3} \sin 1 \le \int_1^2 x^2 \sin x dx \le \frac{7}{3}$$

(c) Let us evaluate $\int_1^2 x^2 \sin x dx$
using $\int_1^2 x^2 \sin x dx \approx \sum_{i=1}^{n} c_i^2 \sin c_i \Delta x$
and $n = 6$
Where $c_i = \dfrac{x_i + x_{i-1}}{2}, x_i = 1 + \dfrac{i}{6}$,
Here $c_i = \dfrac{2 + \frac{i}{6} + \frac{(i-1)}{6}}{2}$
$= \dfrac{(2i+11)}{12}$

$$\sum_{i=1}^{n} \left(c_i^2 \sin c_i\right) \left(\frac{1}{n}\right) = \left[\left(\frac{13}{12}\right)^2 \sin\left(\frac{13}{12}\right) + \left(\frac{15}{12}\right)^2 \sin\left(\frac{15}{12}\right) + \left(\frac{17}{12}\right)^2 \sin\left(\frac{17}{12}\right) + \left(\frac{19}{12}\right)^2 \sin\left(\frac{19}{12}\right) + \left(\frac{21}{12}\right)^2 \sin\left(\frac{21}{12}\right) + \left(\frac{23}{12}\right)^2 \sin\left(\frac{23}{12}\right) \right] \cdot \frac{1}{6}$$

Therefore, $\int_1^2 x^2 \sin x dx \approx 2.2465$

$$(\cos 1 - \cos 2) \le \int_1^2 x^2 \sin x dx \le 4(\cos 1 - \cos 2)$$

$\Rightarrow 0.9564 \le 2.2465 \le 3.8257$
and

$$\frac{7}{3} \sin 1 \le \int_1^2 x^2 \sin x dx \le \frac{7}{3}$$

$\Rightarrow 1.9634 \le 2.2465 \le 2.3333$
The second inequality gives a range which is more closer to the value of the integral. Therefore, part **(b)** is more useful than part **(a).**

42. Notice that $x^2 e^{-\sqrt{x}}$ is a continuous function for all values of $x \ge 0$.
For $1 \le x \le 2$,
$e^{-\sqrt{2}} \le e^{-\sqrt{x}} \le e^{-1}$

Therefore $x^2e^{-\sqrt{2}} \le x^2e^{-\sqrt{x}} \le x^2e^{-1}$

Thus, on using theorem 4.3.

$$\int_1^2 x^2e^{-\sqrt{2}}dx \le \int_1^2 x^2e^{-\sqrt{x}}dx \le \int_1^2 x^2e^{-1}dx$$

$$e^{-\sqrt{2}}\frac{x^3}{3}\bigg|_1^2 \le \int_1^2 x^2e^{-\sqrt{x}}dx \le e^{-1}\frac{x^3}{3}\bigg|_1^2$$

$$\frac{7}{3}e^{-\sqrt{2}} \le \int_1^2 x^2e^{-\sqrt{x}}dx \le \frac{7}{3}e^{-1}$$

$$0.5672 \le \int_1^2 x^2e^{-\sqrt{x}}dx \le 0.8583$$

43. This is just a restatement of the Integral Mean Value Theorem.

44. Let $c = \dfrac{a+b}{2}$. By definition,

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^n f(c_i)\Delta x.$$

We can choose n to be always even, so that $n = 2m$, and

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^n f(c_i)\Delta x$$

$$= \lim_{m\to\infty}\sum_{i=1}^m f(c_i)\Delta x + \lim_{m\to\infty}\sum_{i=m+1}^n f(c_i)\Delta x$$

$$= \int_a^c f(x)dx + \int_c^b f(x)dx$$

45. Between $x = 0$ and $x = 2$, the area below the x-axis is much less than the area above the x-axis. Therefore $\int_0^2 f(x)dx > 0$

46. Between $x = 0$ and $x = 2$, the area above the x-axis is much greater than the area below the x-axis. Therefore $\int_0^2 f(x)dx > 0$

47. Between $x = 0$ and $x = 2$, the area below the x-axis is slightly greater than the area above the x-axis. Therefore $\int_0^2 f(x)dx < 0$

48. Between $x = 0$ and $x = 2$, the area below the x-axis is much greater than the area above the x-axis. Therefore$\int_0^2 f(x)dx < 0$

49. $\displaystyle\int_0^2 3xdx = \frac{1}{2}bh = \frac{1}{2}(2)(6) = 6$

50. $\displaystyle\int_1^4 2xdx = \frac{1}{2}(a+b)h = \frac{1}{2}(2+8)(3)$
$= 15$

51. $\displaystyle\int_0^2 \sqrt{4-x^2} = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi\left(2^2\right) = \pi$

52. $\displaystyle\int_{-3}^0 \sqrt{9-x^2}dx = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi 3^2$
$\displaystyle = \frac{9\pi}{4}$

53. (a) Given limit

$$\lim_{n\to\infty}\frac{1}{n}\left[\sin\left(\frac{\pi}{n}\right) + ... + \sin\left(\frac{n\pi}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\sum_{i=1}^n \sin\left(\frac{i\pi}{n}\right)\right]$$

We know that

$$\lim_{x\to\infty}\left[\sum_{i=1}^n f(c_i)\Delta x\right] = \int_a^b f(x)dx$$

Where $c_i = a + i\Delta x$ and $\Delta x = \left(\dfrac{b-a}{n}\right)$

On comparision,we get

$c_i = \dfrac{i}{n}$, $\Delta x = \dfrac{1}{n}$ and

$f(x) = \sin(\pi x) \Rightarrow a = 0, b = 1$

Therefore

$$\lim_{n\to\infty}\frac{1}{n}\left[\sum_{i=1}^n \sin\left(\frac{i\pi}{n}\right)\right] = \int_0^1 \sin(\pi x)dx$$

(b) Given limit

$$= \lim_{n\to\infty}\left[\frac{n+1}{n^2} + \frac{n+2}{n^2} + ... + \frac{2n}{n^2}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\sum_{i=1}^n \frac{n+i}{n}\right]$$

We know that

$$\lim_{x\to\infty}\left[\sum_{i=1}^n f(c_i)\Delta x\right] = \int_a^b f(x)dx$$

Where $c_i = a + i\Delta x$ and $\Delta x = \left(\dfrac{b-a}{n}\right)$

On comparision,we get

$c_i = \dfrac{i}{n}$, $\Delta x = \dfrac{1}{n}$ and $f(x) = 1 + x$

$\Rightarrow a = 0, b = 1$

Therefore,

$$\lim_{n\to\infty}\frac{1}{n}\left[\sum_{i=1}^n \frac{n+i}{n}\Delta x\right] = \int_0^1 (1+x)dx$$

(c) Given limit

$$\lim_{n\to\infty}\left[\frac{f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + ... + f\left(\frac{n}{n}\right)}{n}\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\sum_{i=1}^n f\left(\frac{i}{n}\right)\right]$$

We know that

$$\lim_{x\to\infty}\left[\sum_{i=1}^n f(c_i)\Delta x\right] = \int_a^b f(x)dx$$

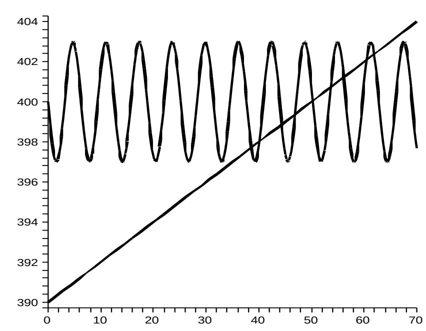
Where $c_i = a + i\Delta x$ and $\Delta x = \left(\dfrac{b-a}{n}\right)$

On comparision,we get
$c_i = \dfrac{i}{n}$ and $\Delta x = \frac{1}{n}$
$\Rightarrow a = 0, b = 1$
Therefore,
$$\lim_{n\to\infty} \frac{1}{n}\left[\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\right] = \int_0^1 f(x)dx$$

54. $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx = v$
$\displaystyle\int_a^b f(x)dx = v(b-a)$
and
$\dfrac{1}{c-b}\displaystyle\int_b^c f(x)dx = w$
$\displaystyle\int_b^c f(x)dx = w(c-b)$
The average value of f over $[a,c]$ is
$\dfrac{1}{c-a}\displaystyle\int_a^c f(x)dx$
$= \dfrac{1}{c-a}\left[\displaystyle\int_a^b f(x)dx + \int_b^c f(x)dx\right]$
$= \dfrac{1}{c-a}[v(b-a) + w(c-b)]$
$= \dfrac{v(b-a) + w(c-b)}{c-a}$

55. Since $b(t)$ represents the birthrate (in births per month), the total number of births from time $t = 0$ to $t = 12$ is given by the integral $\int_0^{12} b(t)\,dt$.
Similarly, the total number of deaths from time $t = 0$ to $t = 12$ is given by the integral $\int_0^{12} a(t)\,dt$.
Of course, the net change in population is the number of birth minus the number of deaths:
Population Change
= Births − Deaths
$= \displaystyle\int_0^{12} b(t)\,dt - \int_0^{12} a(t)\,dt$
$= \displaystyle\int_0^{12} [b(t) - a(t)]\,dt.$
Next we solve the inequality
$410 - 0.3t > 390 + 0.2t$
$20 > 0.5t$ then $t < 40$ months .
Therefore $b(t) > a(t)$ when $t < 40$ months. The population is increasing when the birth rate is greater than the death rate, which is during the first 40 month. After 40 months, the population is decreasing. The population-would reach a maximum at $t = 40$ months.

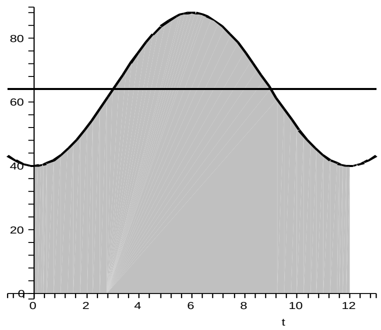
56. Since $b(t)$ represents the birthrate (in births per month), the total number of births from time $t = 0$ to $t = 12$ is given by the integral $\displaystyle\int_0^{12} b(t)dt.$
Similarly, the total number of deaths from time $t = 0$ to $t = 12$ is given by the integral $\displaystyle\int_0^{12} a(t)dt.$
Of course, the net change in population is the number of birth minus the number of deaths:
Population Change
= Births − Deaths
$= \displaystyle\int_0^{12} b(t)dt - \int_0^{12} a(t)dt$
$= \displaystyle\int_0^{12} [b(t) - a(t)]dt.$
By graphing $b(t)$ and $a(t)$ we see that their graphs intersect 9 times, at
$t \approx 38.5, 40.1, 44.4, 46.9, 50.2, 53.6,$
$56.1, 60.5, 61.9.$
This tells us that we have $b(t) > a(t)$ on the intervals
$(0, 38.5), (40.1, 44.4), (46.9, 50.2),$
$(53.6, 56.1), (60.5, 61.9).$
The maximum population will occur when $t = 50.2$.



57. From $PV = 10$ we get $P(V) = 10/V$. By definition,
$\displaystyle\int_2^4 P(V)\,dV = \int_2^4 \frac{10}{V}dV$
$= \displaystyle\sum_{i=1}^{n} \frac{2}{n}\cdot\frac{10}{2+\frac{2i}{n}}$
An estimate of the value of this integral is setting $n = 100$, and then the integral ≈ 6.93

58. The average temperature over the year is
$\dfrac{1}{12}\displaystyle\int_0^{12} 64 - 24\cos\left(\frac{\pi}{6}t\right)\,dt$. If you look at the graphs $T(t)$ and $f(t) = 64$ you should be able to see that the area under $T(t)$ and $f(t)$ between $t = 0$ to $t = 12$ are equal. This means

that the average temperature is 64.



59. Since r is the rate at which items are shipped, rt is the number of items shipped between time 0 and time t. Therefore, $Q - rt$ is the number of items remaining in inventory at time t. Since $Q - rt = 0$ when $t = Q/r$, the formula is valid for $0 \le t \le Q/r$. The average value of $f(t) = Q - rt$ on the time interval $[0, Q/r]$ is

$$\frac{1}{Q/r - 0}\int_0^{Q/r} f(t)dt$$
$$= \frac{r}{Q}\int_0^{Q/r}(Q - rt)dt$$
$$= \frac{r}{Q}\left[Qt - \frac{1}{2}rt^2\right]_0^{Q/r}$$
$$= \frac{r}{Q}\left[\frac{Q^2}{r} - \frac{r}{2}\frac{Q^2}{r^2}\right]$$
$$= \frac{r}{Q}\left[\frac{Q^2}{2r}\right] = \frac{Q}{2}.$$

60. $f(Q) = c_0\dfrac{D}{Q} + c_c\dfrac{Q}{2}$

$f'(Q) = -\dfrac{c_0 D}{Q^2} + \dfrac{c_c}{2}$

Setting $f'(Q) = 0$ gives

$\dfrac{c_0 D}{Q^2} = \dfrac{c_c}{2}$

$Q = \sqrt{\dfrac{2c_0 D}{c_c}}$. This is the right answer of Q minimizing the total cost $f(Q)$, since when the value of Q is very small, the value of D/Q will get very big, and when the value of Q is very small, the value of $Q/2$ will get very big. This means that the function $f(Q)$ is decreasing on the interval $[0, \sqrt{2c_0 D/c_c}]$ and increasing on the interval $[\sqrt{2c_0 D/c_c}, \infty]$. When $Q = \sqrt{2c_0 D/c_c}$,

$$c_0\frac{D}{Q} = \frac{c_0 D}{\sqrt{\frac{2c_0 D}{c_c}}} = c_c\frac{\sqrt{\frac{2c_0 D}{c_c}}}{2} = c_c\frac{Q}{2}.$$

61. Delivery is completed in time Q/p, and since in that time Qr/p items are shipped, the inventory when delivery is completed is

$Q - \dfrac{Qr}{p} = Q\left(1 - \dfrac{r}{p}\right)$.

The inventory at any time is given by

$$g(t) = \begin{cases} (p - r)t & \text{for} \quad t \in \left[0, \frac{Q}{p}\right] \\ Q - rt & \text{for} \quad t \in \left[\frac{Q}{p}, \frac{Q}{r}\right] \end{cases}$$

The graph of g has two linear pieces. The average value of g over the interval $[0, Q/r]$ is the area under the graph (which is the area of a triangle of base Q/r and height $Q(1 - r/p)$) divided by the length of the interval (which is the base of the triangle). Thus the average value of the function is $(1/2)bh$ divided by b, which is

$(1/2)h = (1/2)Q(1 - r/p)$.

This time the total cost is

$f(Q) = c_0\dfrac{D}{Q} + c_c\dfrac{Q}{2}(1 - \dfrac{r}{p})$

$f'(Q) = -\dfrac{c_0 D}{Q^2} + \dfrac{c_c(1 - \frac{r}{p})}{2}$

$f'(Q) = 0$ gives $\dfrac{c_0 D}{Q^2} = \dfrac{c_c}{2}(1 - \dfrac{r}{p})$

$Q = \sqrt{\dfrac{2c_0 D}{c_c(1 - r/p)}}$.

The order size to minimize the total cost is

$Q = \sqrt{\dfrac{2c_0 D}{c_c(1 - r/p)}}$.

62. Use the result from Exercise 60,

$Q = \sqrt{\dfrac{2c_0 D}{c_c}}$

$= \sqrt{\dfrac{2(50,000)(4000)}{3800}} \approx 324.44$.

Since this quantity already takes advantage of largest possible discount, the order size that minimizes the total cost is about 324.44 items.

63. The maximum of

$F(t) = 9 - 10^8(t - 0.0003)^2$

occurs when $10^8(t - 0.0003)^2$ reaches its minimum, that is, when $t = 0.0003$. At that time

$F(0.0003) = 9$ thousand pounds.

We estimate the value of

$\displaystyle\int_0^{0.0006}[9 - 10^8(t - 0.0003)^2]\,dt$ using midpoint sum and $n = 20$, and get $m\Delta v \approx 0.00360$ thousand pound-seconds, so $\Delta v \approx 360$ ft per second.

64. The impulse-momentum equation of Prob-

lem 65 gives $5\Delta v$
$= \int_0^{0.4} (1000 - 25,000(t-0.2)^2)\, dt$
$= \int_0^{0.4} (-25000t^2 + 10000t)\, dt$
Using a midpoint sum and $n = 20$ gives an approximation for this integral of 267.0. This means $5\Delta v \approx 267$ and $\Delta v \approx 53.4$ m/s

4.5 The Fundamental Theorem of Calculus

1. $\int_0^2 (2x-3)dx = \left(x^2 - 3x\right)\Big|_0^2 = -2$

2. $\int_0^3 \left(x^2 - 2\right)dx = \left(\frac{x^3}{3} - 2x\right)\Big|_0^3 = 3$

3. $\int_{-1}^1 \left(x^3 + 2x\right)dx = \left(\frac{x^4}{4} + x^2\right)\Big|_{-1}^1 = 0$

4. $\int_0^2 \left(x^3 + 3x - 1\right)dx$
$= \left(\frac{x^4}{4} - \frac{3x^2}{2} - x\right)\Big|_0^2 = -4$

5. $\int_1^4 \left(x\sqrt{x} + \frac{3}{x}\right)dx$
$= \left(\frac{2}{5}x^{5/2} + 3\log x\right)\Big|_1^4$
$= \frac{2}{5}\cdot 32 + 3\log 4 - \frac{2}{5}.1 - 3\log 1$
$= \frac{62}{5} + 3\log 4$

6. $\int_1^2 \left(4x - \frac{2}{x^2}\right)dx = \left(2x^2 + \frac{2}{x}\right)\Big|_1^2 = 5$

7. $\int_0^1 \left(6e^{-3x} + 4\right)dx = \left(\frac{6e^{-3x}}{-3} + 4x\right)\Big|_0^1$
$= -\frac{2}{e^3} + 4 + 2 - 0 = -\frac{2}{e^3} + 6$

8. $\int_0^2 \left(\frac{e^{2x} - 2e^{3x}}{e^{3x}}\right)dx$
$= \int_0^2 \left(e^{-x} - 2\right)dx = \left(-e^{-x} - 2x\right)\Big|_0^2$
$= -\frac{1}{e^2} - 3$

9. $\int_{\pi/2}^{\pi} (2\sin x - \cos x)dx = -2\cos x - \sin x\Big|_{\pi/2}^{\pi}$
$= 3$

10. $\int_{\pi/4}^{\pi/2} 3\csc x\cot x dx = (-3\csc x)\Big|_{\pi/4}^{\pi/2}$
$= -3 + 3\sqrt{2}$

11. $\int_0^{\pi/4} (\sec t\tan t)\, dt = \sec t\Big|_0^{\pi/4}$
$= \sqrt{2} - 1$

12. $\int_0^{\pi/4} \sec^2 t dt = \tan t\Big|_0^{\pi/4} = 1$

13. $\int_0^{1/2} \frac{3}{\sqrt{1-x^2}}dx = 3\sin^{-1}x\Big|_0^{1/2}$
$= 3\left(\frac{\pi}{6} - 0\right) = \frac{\pi}{2}$

14. $\int_{-1}^1 \frac{4}{1+x^2}dx = 4\arctan x\Big|_{-1}^1 = 2\pi$

15. $\int_1^4 \frac{t-3}{t}dt$
$= \int_1^4 \left(1 - 3t^{-1}\right)dt = (t - 3\ln|t|)\Big|_1^4$
$= 3 - 3\ln 4$

16. $\int_0^4 t\,(t-2)\,dt = \left(\frac{t^3}{3} - t^2\right)\Big|_0^4 = \frac{16}{3}$

17. $\int_0^t \left(e^{x/2}\right)^2 dx = (e^x)\Big|_0^t = e^t - 1$

18. $\int_0^t \left(\sin^2 x + \cos^2 x\right)dx$
$= \int_0^t 1dx = (x)\Big|_0^t = t$

19. The graph of $y = 4 - x^2$ is above the x-axis over the interval $[-2, 2]$.
$\int_{-2}^2 (4 - x^2)dx = \left(4x - \frac{x^3}{3}\right)\Big|_{-2}^2 = \frac{32}{3}$

20. The graph of $y = x^2 - 4x$ is below the x-axis over the interval $[0, 4]$.
$\int_0^4 -\left(x^2 - 4x\right)dx = \left(-\frac{x^3}{3} + 2x^2\right)\Big|_0^4 = \frac{32}{3}$

21. The graph of $y = x^2$ is above the x-axis over the interval $[0, 2]$.
$\int_0^2 x^2 dx = \frac{x^3}{3}\Big|_0^2 = \frac{8}{3}$

22. The graph of $y = x^3$ is above the x-axis over the interval
$[0, 3]$.
$$\int_0^3 x^3 dx = \left(\frac{x^4}{4}\right)\Big|_0^3 = \frac{81}{4}$$

23. The graph of $y = \sin x$ is above the x-axis over the interval $[0, \pi]$.
$$\int_0^\pi \sin x dx = -\cos x\Big|_0^\pi = 2$$

24. The graph of $y = \sin x$ is below the x-axis over the interval $\left[-\frac{\pi}{2}, 0\right]$ and above the x-axis over the interval $\left[0, \ \frac{\pi}{4}\right]$. Hence we need to compute two seperate integrals and add them together:
$$\int_{-\pi/2}^0 -\sin x dx + \int_0^{\pi/4} \sin x dx$$
$$= 1 + \left(1 - \frac{1}{\sqrt{2}}\right) = 2 - \frac{1}{\sqrt{2}}.$$

25. $f'(x) = x^2 - 3x + 2$

26. $f'(x) = x^2 - 3x - 4$

27. $f'(x) = \left(e^{-\left(x^2\right)^2} + 1\right) \frac{d}{dx}\left(x^2\right)$
$= 2x\left(e^{-x^4} + 1\right)$

28. $f'(x) = -\sec x$

29. $f(x) = \int_{e^x}^0 \sin t^2 dt + \int_0^{2-x} \sin t^2 dt$
$$f'(x) = -\sin e^{2x} \frac{d}{dx}\left(e^x\right)$$
$$+ \sin(2-x)^2 \frac{d}{dx}(2-x)$$
$$= -e^x \sin e^{2x} - \sin(2-x)^2$$

30. $f(x) = \int_{2-x}^0 e^{2t} dt + \int_0^{xe^x} e^{2t} dt$
$$f'(x) = -e^{2(2-x)} \frac{d}{dx}(2-x)$$
$$+ e^{2(xe^x)} \frac{d}{dx}\left(xe^x\right)$$
$$= e^{4-2x} + e^{2xe^x}\left(xe^x + e^x\right)$$

31. $f(x) = \int_{x^2}^0 \sin(2t)\, dt + \int_0^{x^3} \sin(2t)\, dt$
$$f'(x) = -\sin\left(2x^2\right) \frac{d}{dx}\left(x^2\right)$$
$$+ \sin\left(2x^3\right) \frac{d}{dx}\left(x^3\right)$$
$$= -2x\sin\left(2x^2\right) + 3x^2\sin\left(2x^3\right)$$

32. $f(x) == \int_{3x}^0 \left(t^2 + 4\right) dt + \int_0^{\sin x} \left(t^2 + 4\right) dt$
$$= -\int_0^{3x} \left(t^2 + 4\right) dt + \int_0^{\sin x} \left(t^2 + 4\right) dt$$
$$f'(x) = -\left(9x^2 + 4\right) \frac{d}{dx}(3x)$$
$$+ \left(\sin^2 x + 4\right) \frac{d}{dx}(\sin x)$$
$$= -27x^2 - 12 + \sin^2 x \cos x + 4\cos x$$

33. $s(t) = 40t + \cos t + c$,
$s(0) = 0 + \cos 0 + c = 2$
so therefore $c = 1$ and $s(t) = 40t + \cos t + 1$.

34. $s(t) = 10e^t + c$,
$s(0) = 10 + c = 2$
so therefore $c = -8$ and $s(t) = 10e^{-t} - 8$.

35. $v(t) = 4t - \frac{t^2}{2} + c_1$,
$v(0) = c_1 = 8$
so therefore $c_1 = 8$ and $v(t) = 4t - \frac{t^2}{2} + 8$.
$s(t) = 2t^2 - \frac{t^3}{6} + 8t + c_2$,
$s(0) = c_2 = 0$
so therefore $c_2 = 0$ and $s(t) = 2t^2 - \frac{t^3}{6} + 8t$.

36. $v(t) = 16t - \frac{t^3}{3} + c_1$,
$v(0) = c_1 = 0$
so therefore $c_1 = 0$ and
$v(t) = 16t - \frac{t^3}{3}$.
$s(t) = 8t^2 - \frac{t^4}{12} + c_2$,
$s(0) = c_2 = 30$
so therefore $c_2 = 30$ and $s(t) = 8t^2 - \frac{t^4}{12} + 30$.

37. Let $w(t)$ be the number of gallons in the tank at time t.

(a) The water level decreases if $w'(t) = f(t) < 0$ i.e. if $f(t) = 10\sin t < 0$, for which $\pi < t < 2\pi$.
Alternatively, the water level increases if $w'(t) = f(t) > 0$ i.e. if $f(t) = 10\sin t > 0$, for which $0 < t < \pi$.

(b) Now,we start with
$w'(t) = 10\sin t$
Therefore, $\int_0^\pi w'(t)\, dt = \int_0^\pi 10\sin t dt$
$w(\pi) - w(0) = -10\cos t|_0^\pi$

But $w(0) = 100$.
Therefore,
$w(\pi) - 100 = -10(-1-1) = 20$
$\Rightarrow w(\pi) = 120$.
Therefore the tank will have 120 gallons at $t = \pi$.

38. Let $w(t)$ be the number of thousand gallons in the pond at time t.

(a) The water level decreases if $w'(t) = f(t) < 0$ i.e. if $f(t) = 4t - t^2 < 0$, for which $4 < t \leq 6$.
Alternatively, the water level increases if $w'(t) = f(t) > 0$ i.e. if $f(t) = 4t - t^2 > 0$, for which $0 < t < 4$.

(b) Now, we start with $w'(t) = 4t - t^2$, Therefore
$$\int_0^6 w'(t)\,dt = \int_0^6 \left(4t - t^2\right) dt$$
$$w(6) - w(0) = \left(2t^2 - \frac{t^3}{3}\right)\Bigg|_0^6$$
But $w(0) = 40$.
Therefore,
$w(6) - 40 = 72 - 72 = 0$
$\Rightarrow w(6) = 40$.
Therefore the pond has 40,000 gallons at t=6.

39. $y'(x) = \sin\sqrt{x^2 + \pi^2}$.
At the point in question, $y(0) = 0$ and $y'(0) = \sin\pi = 0$. Therefore, the tangent line has slope 0 and passes through the point $(0,\ 0)$. The equation of this line is $y = 0$.

40. $y'(x) = \ln\left(x^2 + 2x + 2\right)$.
At the point in question, $y(-1) = 0$ and $y'(-1) = \ln 1 = 0$. Therefore, the tangent line has slope 0 and passes through the point $(-1,\ 0)$. The equation of this line is $y = 0$.

41. $y'(x) = \cos\left(\pi x^3\right)$.
At the point in question, $y(2) = 0$ and $y'(2) = \cos 8\pi = 1$. Therefore, the tangent line has slope 1 and passes through the point $(2,\ 0)$. The equation of this line is $y = x - 2$.

42. $y'(x) = e^{-x^2+1}$.
At the point in question, $y(0) = 0$ and $y'(0) = e$. Therefore, the tangent line has slope e and passes through the point $(0,\ 0)$. The equation of this line is $y = ex$.

43. $$\int_0^2 \sqrt{x^2+1}dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\sqrt{\left(\frac{2i}{n}+1\right)}$$
Estimating using $n = 20$, we get the Riemann sum ≈ 2.96.

44. $$\int_0^2 \left(\sqrt{x}+1\right)^2 dx = \int_0^2 \left(x + 2\sqrt{x} + 1\right)dx$$
$$= \left(\frac{x^2}{2} + \frac{4}{3}x^{\frac{3}{2}} + x\right)\Bigg|_0^2 = 4 + \frac{8\sqrt{2}}{3}.$$

45. $$\int_1^4 \frac{x^2}{x^2+4}dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\frac{\left(1+(3i/n)^2\right)}{(3i/n)^2+4}$$
Estimating using $n = 20$, we get the Riemann sum ≈ 1.71.

46. $$\textstyle\int_1^4 \frac{x^2+4}{x^2}dx = \int_1^4 1 + \frac{4}{x^2}dx = \left(x - 4x^{-1}\right)\Big|_1^4 = 6$$

47. $$\int_0^{\pi/4} \frac{\sin x}{\cos^2 x}dx$$
$$= \int_0^{\pi/4} \tan x \sec x dx = \sec x\Bigg|_0^{\pi/4} = \sqrt{2} - 1$$

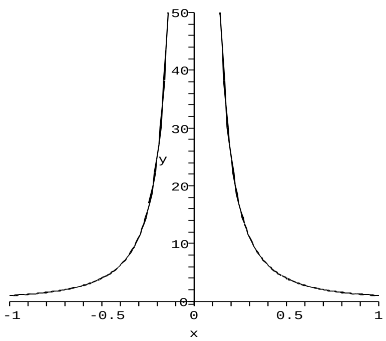
48. $$\int_0^{\pi/4} \frac{\tan x}{\sec^2 x}dx = \int_0^{\pi/4} \sin x \cos x dx$$
$$= \int_0^{\pi/4} \frac{1}{2}\sin 2x dx = \left(-\frac{1}{4}\cos 2x\right)\Bigg|_0^{\pi/4} = \frac{1}{4}$$

49. From the graph of $f(x)$,
$$\int_0^3 f(x)\,dx < \int_0^2 f(x)\,dx < \int_0^1 f(x)\,dx.$$
The function increases if $g'(x) = f(x) > 0$ i.e. when $x < 1$ or $x > 3$. Thus, the function $g(x)$ is increasing in the intervals $(-\infty, 1)$ and $(3, \infty)$. The function $g(x)$ has critical points at $g'(x) = 0$. i.e. when $x = 1$ or $x = 3$. Therefore the critical points of $g(x)$ are $x = 1$ and $x = 3$.

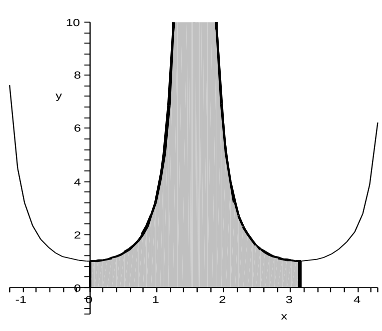
50. $$\int_0^1 f(x)\,dx < \int_0^3 f(x)\,dx < \int_0^2 f(x)\,dx.$$
The function increases if $g'(x) = f(x) > 0$ i.e. when $0 < x < 2$ or $x > 4$. Thus, the function $g(x)$ is increasing in the intervals $(0, 2)$ and $(4, \infty)$. The function $g(x)$ has critical points at $g'(x) = 0$ i.e.when $x = 0$, $x = 2$ and $x = 4$. Therefore the critical points of $g(x)$ are $x = 0$, $x = 2$ and $x = 4$.

51. If you look at the graph of $1/x^2$, it is obvious that there is positive area between the curve and the x-axis over the interval $[-1, 1]$. In addition to this, there is a vertical asymptote in the interval that we are integrating over which should alert us to a possible problem.

The problem is that $1/x^2$ is not continuous on $[-1, 1]$ (the discontinuity occurs at $x = 0$) and that continuity is one of the conditions in the Fundamental Theorem of Calculus, Part I(Theorem 4.1).



52. If you look at the graph of $\sec^2 x$, it is obvious that there is positive area between the curve and the x-axis over the interval $[0, \pi]$. In addition to this, there is a vertical asymptote in the interval that we are integrating over which should alert us to a possible problem. The problem is that $\sec^2 x$ is not continuous on $[0, \pi]$ and that continuity is one of the conditions in the Fundamental Theorem of Calculus, Part I (Theorem 4.1).



53. The integrals in parts (a) and (c) are improper, because the integrands have asymptotes at one of the limits of integration. The Fundamental Theorem of Calculus applies to the integral in part (b).

54. The Fundamental Theorem of Calculus applies to the integral in part (a). The integral in part (b) is improper since the point $x = 3$ lies in the interval $[0, 4]$, and $\dfrac{1}{(x-3)^2}$ is not defined at $x = 3$. The integral in part (c) is improper since the point $x = \pi/2$ lies in the interval $[0, 2]$, and $\sec x$ is not defined at $x = \pi/2$.

55. $$f_{ave} = \frac{1}{3-1}\int_1^3 \left(x^2 - 1\right)dx = \frac{1}{2}\left(\frac{x^3}{3} - x\right)\Big|_1^3 = \frac{10}{3}$$

56. $$f_{ave} = \frac{1}{1-0}\int_0^1 \left(2x - 2x^2\right)dx = \left(x^2 - \frac{2x^3}{3}\right)\Big|_0^1 = \frac{1}{3}$$

57. $$f_{ave} = \frac{1}{\pi/2 - 0}\int_0^{\pi/2} \cos x dx = \frac{2}{\pi}\left(\sin x\right)\Big|_0^{\pi/2} = \frac{2}{\pi}$$

58. $$f_{ave} = \frac{1}{2-0}\int_0^2 e^x dx = \frac{1}{2}\left(e^x\right)\Big|_0^2 = \frac{1}{2}\left(e^2 - 1\right)$$

59. (a) Using the Fundamental Theorem of Calculus, it follows that an antiderivative of e^{-x^2} is $\int_a^x e^{-t^2}dt$ where a is a constant.

(b) Using the Fundamental Theorem of Calculus, it follows that an antiderivative of $\sin\sqrt{x^2+1}$ is $\int_a^x \sin\sqrt{t^2+1}dt$ where a is a constant.

60. It may be observed that f is piecewise continuous over its domain.
For $0 < x \le 4$,
$$g(x) = \int_0^x f(t)\,dt = \int_0^x \left(t^2 + 1\right)dt = \left(\frac{t^3}{3} + t\right)\Big|_0^x = \frac{x^3}{3} + x$$
Now, for $x > 4$
$$\begin{aligned} g(x) &= \int_0^x f(t)\,dt \\ &= \int_0^4 f(t)\,dt + \int_4^x f(t)\,dt \\ &= \int_0^4 \left(t^2+1\right)dt + \int_4^x \left(t^3 - t\right)dt \\ &= \left(\frac{t^3}{3} + t\right)\Big|_0^4 + \left(\frac{t^4}{4} - \frac{t^2}{2}\right)\Big|_4^x \end{aligned}$$

$$= \left(\frac{4^3}{3}+4\right)+\left(\frac{x^4}{4}-\frac{x^2}{2}-\frac{4^4}{4}+\frac{4^2}{2}\right)$$
$$= \frac{x^4}{4}-\frac{x^2}{2}-\frac{92}{3}$$
$$g(x) = \begin{cases} \frac{x^3}{3}+x & \text{for } 0<x\le 4 \\ \frac{x^4}{4}-\frac{x^2}{2}-\frac{92}{3} & \text{for } 4<x \end{cases}$$

Consider

$$g'(4) = \lim_{h\to 0}\frac{g(4+h)-g(4)}{h}$$
$$= \lim_{h\to 0}\frac{1}{h}\left[\int_0^{4+h} f(t)\,dt - \int_0^4 f(t)\,dt\right]$$
$$= \lim_{h\to 0}\frac{1}{h}\int_4^{4+h} f(t)\,dt.$$

The Right Hand Limit:

$$\lim_{h\to 0^+}\frac{1}{h}\int_4^{4+h} f(t)\,dt$$
$$= \lim_{h\to 0^+}\frac{1}{h}\int_4^{4+h}\left(t^3-t\right)dt$$
$$= \lim_{h\to 0^+}\frac{1}{h}\left[\frac{t^4}{4}-\frac{t^2}{2}\right]_4^{4+h}$$
$$= \lim_{h\to 0^+}\frac{1}{h}\left[\frac{(4+h)^4}{4}-\frac{(4+h)^2}{2}-\frac{4^4}{4}+\frac{4^2}{2}\right]$$
$$= \lim_{h\to 0^+}\frac{1}{h}\left[\frac{h^4}{4}+4h^3-\frac{47h^2}{2}+60h\right]$$
$$= \lim_{h\to 0^+}\left[\frac{h^3}{4}+4h^2-\frac{47h}{2}+60\right]=60.$$

Now, the Left Hand Limit:

$$\lim_{h\to 0^-}\frac{1}{h}\int_4^{4+h} f(t)\,dt$$
$$= \lim_{h\to 0^-}\frac{1}{h}\int_4^{4+h}\left(t^2+1\right)dt$$
$$= \lim_{h\to 0^-}\frac{1}{h}\left[\frac{t^3}{3}+t\right]_4^{4+h}$$
$$= \lim_{h\to 0^-}\frac{1}{h}\left[\frac{(4+h)^3}{3}+4+h-\frac{4^3}{3}-4\right]$$
$$= \lim_{h\to 0^+}\frac{1}{h}\left[\frac{h^3+12h^2+48h+64}{3}+h-\frac{64}{3}\right]$$
$$= \lim_{h\to 0^+}\left[\frac{h^2}{3}+4h+17\right]=17.$$

Therefore, $g'(4)$ doesn't exist though $f(4)$ exists. Therefore $g'(x)=f(x)$ is not true for all $x\ge 0$.

61. $f'(x)=x^2-3x+2.$
Setting $f'(x)=0$, we get $(x-1)(x-2)=0$ which implies $x=1, 2$.

$$f'(x) = \begin{cases} >0 & \text{when } t<1 \text{ or } t>2 \\ <0 & \text{when } 1<t<2 \end{cases}$$

$$f(1) = \int_0^1\left(t^2-3t+2\right)dt$$
$$= \left(\frac{t^3}{3}-\frac{3t^2}{2}+2t\right)\Big|_0^1 = \frac{5}{6}$$
$$f(2) = \int_0^2\left(t^2-3t+2\right)dt$$
$$= \left(\frac{t^3}{3}-\frac{3t^2}{2}+2t\right)\Big|_0^2 = \frac{2}{3}$$

Hence $f(x)$ has a local maximum at the point $\left(1,\frac{5}{6}\right)$ and local minimum at the point $\left(2,\frac{2}{3}\right)$.

62. $g(x)=\int_0^x\left[\int_0^u f(t)\,dt\right]du$

$$g'(x)=\int_0^x f(t)\,dt$$
$$g''(x)=f(x)$$

A zero of f corresponds to a zero of the second derivative of g (possibly an inflection point of g).

63. When $a<2$ or $a>2$, f is continuous. Using the Fundamental Theorem of Calculus,

$$\left[\lim_{x\to a}F(x)\right]-F(a)$$
$$= \lim_{x\to a}\left[F(x)-F(a)\right]$$
$$= \lim_{x\to a}\left[\int_0^x f(t)\,dt-\int_0^a f(t)\,dt\right]$$
$$= \lim_{x\to a}\left[\int_a^x f(t)\,dt\right]=0$$

When $a=2$,

$$\lim_{x\to a^-}\left[\int_a^x f(t)\,dt\right]=\lim_{x\to 2^-}\left[\int_2^x t\,dt\right]$$
$$= \lim_{x\to 2^-}\left[\frac{t^2}{2}\right]_2^x=\lim_{x\to 2^-}\left[\frac{x^2}{2}-\frac{2^2}{2}\right]=0$$

and $\lim_{x\to a^+}\left[\int_a^x f(t)\,dt\right]$

$$= \lim_{x\to 2^+}\left[\int_2^x (t+1)\,dt\right]$$
$$= \lim_{x\to 2^+}\left[\frac{t^2}{2}+t\right]_2^x$$
$$= \lim_{x\to 2^+}\left[\frac{x^2}{2}+x-\frac{2^2}{2}-2\right]$$
$$=0$$

Thus, for all value of a,

$$\left[\lim_{x\to a}F(x)\right]-F(a)=0$$
$$\lim_{x\to a}F(x)=F(a)$$

Thus, F is continuous for all x. However, $F'(2)$ does not exist, which is shown as follows:

$$F'(2)=\lim_{h\to 0}\frac{F(2+h)-F(2)}{h}$$

$= \lim_{h\to 0} \frac{1}{h}\left[\int_0^{2+h} f(t)\,dt - \int_0^2 f(t)\,dt\right]$
$= \lim_{h\to 0} \frac{1}{h}\int_2^{2+h} f(t)\,dt$
We will show that this limit does not exist by showing that the left and right limits are different. The right limit is
$\lim_{h\to 0^+} \frac{1}{h}\int_2^{2+h} f(t)\,dt$
$= \lim_{h\to 0^+} \frac{1}{h}\int_2^{2+h} (t+1)\,dt$
$= \lim_{h\to 0^+} \frac{1}{h}\left[\frac{t^2}{2}+t\right]_2^{2+h}$
$= \lim_{h\to 0^+} \frac{1}{h}\left[\frac{(2+h)^2}{2}+2+h-\frac{2^2}{2}-2\right]$
$= \lim_{h\to 0^+} \frac{1}{h}\left[\frac{h^2+4h+4}{2}+2+h-4\right]$
$= \lim_{h\to 0^+} \frac{1}{h}\left[\frac{h^2}{2}+3h\right]$
$= \lim_{h\to 0^+} \frac{1}{h}\left[\frac{h}{2}+3\right] = 3$
The left limit is
$\lim_{h\to 0^-} \frac{1}{h}\int_2^{2+h} f(t)\,dt$
$= \lim_{h\to 0^-} \frac{1}{h}\int_2^{2+h} tdt$
$= \lim_{h\to 0^-} \frac{1}{h}\left[\frac{t^2}{2}\right]_2^{2+h}$
$= \lim_{h\to 0^-} \frac{1}{h}\left[\frac{(2+h)^2}{2}-\frac{2^2}{2}\right]$
$= \lim_{h\to 0^-} \frac{1}{h}\left[\frac{h^2+4h+4}{2}-2\right]$
$= \lim_{h\to 0^-} \frac{1}{h}\left[\frac{h}{2}+2\right] = 2$
Thus, $F'(2)$ does not exist. This result does not contradict the Fundamental Theorem of Calculus, because in this situation, $f(x)$ is not continuous, and thus The Fundamental Theorem of Calculus does not apply.

64. When $x = 0$,
$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f(x^n)$
$= \lim_{n\to\infty} f(0) = f(0)$
When $0 < x < 1$,
$\lim_{n\to x^n} = 0$, and then
$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f(x^n)$
$= f\left(\lim_{n\to\infty} x^n\right) = f(0)$
$= \lim_{n\to\infty} f(0) = f(0)$
When $x = 1$,
$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f(x^n)$
$= \lim_{n\to\infty} f(1) = f(1)$.
Thus the integral $\int_0^1 g_n(x)\,dx$ represents the net area between the graph of $f(x^n)$ and the x-axis. As n approaches ∞,
$f(x^n) \to \begin{cases} f(0) & \text{when } 0 \le x < 1 \\ f(1) & \text{when } x = 1 \end{cases}$
Thus the integral $\int_0^1 g_n(x)\,dx$ approaches the area of the shape of a rectangle with length 1 and width $f(0)$ (possibly negative), which means $\lim_{n\to\infty} \int_0^1 g_n(x)\,dx = f(0)$.

65. $\int_0^x [f(t) - g(t)]\,dt$
$= \int_0^x [55 + 10\cos t - (50+2t)]\,dt$
$= \int_0^x [5 + 10\cos t - 2t]\,dt$
$= 5t + \sin t - t^2\big|_0^x$
$= 5x + \sin x - x^2$
Since we are integrating the difference in speeds, the integral represents the distance that Katie is ahead at time x. Of course, if this value is negative, is means that Michael is really ahead.

66. (a) $CS = \int_0^Q D(q)\,dq - PQ$
$= \int_0^Q \left(150q - 2q - 3q^2\right)dq - PQ$
$= \left(150q - q^2 - q^3\right)\big|_0^Q - PQ$
$= 150Q - Q^2 - Q^3$
$- \left(150 - 2Q - 3Q^2\right)Q$
$= Q^2 + 2Q^3$.
When $Q = 4$,
$CS = 16 + 2(64) = 144$ dollors
When $Q = 6$, $CS = 36 + 2(216) =$ 468 dollors
The consumer surplus is higher for $Q = 6$ than that for $Q = 4$.

(b) $CS = \int_0^Q D(q)\,dq - PQ$
$= \int_0^Q 40e^{-0.05q}dq - PQ$
$= \left(-800e^{-0.05q}\right)\big|_0^Q - PQ$
$= -800e^{-0.05Q} + 800 - 40e^{-0.05Q}$
$= -840e^{-0.05Q} + 800$.
When $Q = 10$, $CS = -840e^{-0.5} + 800 \approx$ 290.5 dollors
When $Q = 20$, $CS = -840e^{-1} + 800 \approx$ 491.0 dollors

The consumer surplus is higher for $Q = 20$ than that for $Q = 10$.

67. The next shipment must arrive when the inventory is zero. This occurs at time $T : f(t) = Q - r\sqrt{t}$
$f(T) = 0 = Q - r\sqrt{T}$
$r\sqrt{T} = Q$
$T = \frac{Q^2}{r^2}$
The average value of f on [0, T] is
$\frac{1}{T}\int_0^T f(t)\,dt$
$= \frac{1}{T}\int_0^T \left(Q - rt^{1/2}\right) dt$
$= \frac{1}{T}\left[Qt - \frac{2}{3}rt^{3/2}\right]_0^T$
$= \frac{1}{T}\left[QT - \frac{2}{3}rT^{3/2}\right]$
$= Q - \frac{2}{3}r\sqrt{T}$
$= Q - \frac{2}{3}r\frac{Q}{r}$
$= \frac{Q}{3}$

68. The total annual cost $f(Q) = c_0\frac{D}{Q} + c_cA = c_0\frac{D}{Q} + c_c\frac{Q}{3}$
$f'(Q) = -c_0\frac{D}{Q^2} + c_c\frac{1}{3}$
$f'(Q) = 0$
gives that $Q = \sqrt{\frac{3c_0D}{c_c}}$.
This value of Q minimizes the total cost, since

$$f'(Q)\begin{cases} > 0 & \text{when} \quad Q < \sqrt{\frac{3c_0D}{c_c}} \\ < 0 & \text{when} \quad Q > \sqrt{\frac{3c_0D}{c_c}} \end{cases}$$

When $Q = \sqrt{\frac{3c_0D}{c_c}}$,
$c_0\frac{D}{Q} = c_0\frac{D}{\sqrt{3c_0D/c_c}} = c_c\frac{Q}{3} = c_cA$

4.6 Integration By Substitution

1. Let $u = x^3 + 2$ and then $du = 3x^2dx$ and
$\int x^2\sqrt{x^3+2}dx = \frac{1}{3}\int u^{-1/2}du$
$= \frac{2}{9}u^{3/2} + c = \frac{2}{9}(x^3+2)u^{3/2} + c.$

2. Let $u = x^4 + 1$and then $du = 4x^3dx$ and
$\int x^3(x^4+1)^{-2/3}dx = \frac{1}{4}\int u^{-2/3}du$
$= \frac{3}{4}u^{1/3} + c = \frac{3}{4}(x^4+1)^{1/3} + c.$

3. Let $u = \sqrt{x} + 2$ and then $du = \frac{1}{2}x^{-1/2}dx$ and
$\int \frac{(\sqrt{x}+2)^3}{\sqrt{x}}dx = 2\int u^3du$
$= \frac{2}{4}u^4 + c = \frac{1}{2}(\sqrt{x}+2)^4 + c.$

4. Let $u = \sin x$and then $du = \cos x dx$ and
$\int \sin x\cos x dx = \int u du$
$= \frac{u^2}{2} + c = \frac{\sin^2 x}{2} + c.$

5. Let $u = x^4 + 3$ and then $du = 4x^3dx$ and
$\int x^3\sqrt{x^4+3}dx = \frac{1}{4}\int u^{1/2}du$
$= \frac{1}{6}u^{3/2} + c = \frac{1}{6}(x^4+3)^{3/2} + c.$

6. Let $u = 1 + 10x$, and then $du = 10dx$ and
$\int \sqrt{1+10x}dx = \frac{1}{10}\int \sqrt{u}du$
$= \frac{1}{10}\int u^{1/2}du = \frac{1}{15}u^{3/2} + c$
$= \frac{1}{15}(1+10x)^{3/2} + c.$

7. Let $u = \cos x$ and then $du = -\sin x dx$ and
$\int \frac{\sin x}{\sqrt{\cos x}}dx = -\int \frac{du}{\sqrt{u}}$
$= -2\sqrt{u} + c = -2\sqrt{\cos x} + c.$

8. Let $u = \sin x$ and then $du = \cos x dx$ and
$\int \sin^3 x\cos x dx = \int u^3du$
$= \frac{u^4}{4} + c = \frac{\sin^4 x}{4} + c.$

9. Let $u = t^3$ and then $du = 3t^2dt$ and
$\int t^2\cos t^3dt = \frac{1}{3}\int \cos u du$
$= \frac{1}{3}\sin u + c = \frac{1}{3}\sin t^3 + c$

10. Let $u = \cos t + 3$ and then $du = -\sin t dt$ and
$\int \sin t(\cos t+3)^{3/4}dt = -\int u^{3/4}du$
$= -\frac{4}{7}u^{7/4} + c = -\frac{4}{7}(\cos t+3)^{7/4} + c.$

11. Let $u = x^2 + 1$ and then $du = 2xdx$ and
$\int xe^{x^2+1}dx = \int \frac{1}{2}e^u du = \frac{1}{2}e^u + c$
$= \frac{1}{2}e^{x^2+1} + c$

12. Let $u = e^x + 4$ and then $du = e^x dx$ and
$\displaystyle\int e^x\sqrt{e^x+4}dx = \int \sqrt{u}du = \frac{2}{3}u^{3/2} + c$
$= \frac{1}{2}(e^x+4)^{3/2} + c$

13. Let $u = \sqrt{x}$ and then $du = \frac{1}{2\sqrt{x}}dx$ and
$\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = 2\int e^u du = 2e^u + c = 2e^{\sqrt{x}} + c$

14. Let $u = \dfrac{1}{x}$ and then $du = -\dfrac{1}{x^2}dx$ and
$\displaystyle\int \frac{\cos\left(\frac{1}{x}\right)}{x^2}dx = -\int \cos u du = -\sin u + c$
$= -\sin\dfrac{1}{x} + c$

15. Let $u = \ln x$ and then $du = \frac{1}{x}dx$ and
$\displaystyle\int \frac{\sqrt{\ln x}}{x}dx = \int \sqrt{u}du = \frac{2}{3}u^{3/2} + c$
$= \dfrac{2}{3}(\ln x)^{3/2} + c$

16. Let $u = \tan x$ and then $du = \sec^2 x dx$ and
Let $u = \ln x$ and then $du = \frac{1}{x}dx$ and
$\displaystyle\int \sec^2 x\sqrt{\tan x}dx = \int u^{1/2}du$
$= \dfrac{2}{3}u^{3/2} + c = \dfrac{2}{3}(\sqrt{\tan x})^{3/2} + c$

17. Let $t = \sqrt{u} + 1$ and then
$dt = \dfrac{1}{2}u^{-1/2}du = \dfrac{1}{2\sqrt{u}}du$ and
$\displaystyle\int \frac{1}{\sqrt{u}\left(\sqrt{u}+1\right)}du = 2\int \frac{1}{t}dt = 2\ln|t| + c$
$= 2\ln\left|\sqrt{u}+1\right| + c = 2\ln\left(\sqrt{u}+1\right) + c$

18. Let $u = v^2 + 4$ and then $du = 2vdv$ and
$\displaystyle\int \frac{v}{v^2+4}dv = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + c$
$= \dfrac{1}{2}\ln\left|v^2+4\right| + c = \dfrac{1}{2}\ln\left(v^2+4\right) + c$

19. Let $u = \ln x + 1$ and then $du = \dfrac{1}{x}dx$ and
$\displaystyle\int \frac{4}{x(\ln x+1)^2}dx = 4\int u^{-2}du$
$= -4u^{-1} + c = -4(\ln x + 1)^{-1} + c$

20. Let $u = \cos 2x$ and then $du = -2\sin 2x dx$ and
$\displaystyle\int \tan 2x dx = -\frac{1}{2}\int \frac{1}{u}du$
$= -\dfrac{1}{2}\ln|u| + c = -\dfrac{1}{2}\ln|\cos 2x| + c$

21. Let $u = \sin^{-1}x$ and then $du = \dfrac{1}{\sqrt{1-x^2}}dx$ and
Let $u = \cos 2x$ and then $du = -2\sin 2x dx$ and
$\displaystyle\int \frac{\left(\sin^{-1}x\right)^3}{\sqrt{1-x^2}}dx = \int u^3 du$
$= \dfrac{u^4}{4} + c = \dfrac{\left(\sin^{-1}x\right)^4}{4} + c$

22. Let $u = x^2$ and then $du = 2xdx$ and
$\displaystyle\int \frac{x}{\sqrt{1-x^4}}dx = \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}du$
$= \dfrac{1}{2}\sin^{-1}u + c = \dfrac{1}{2}\sin^{-1}x^2 + c$

23. (a) Let $u = x^2$ and then $du = 2xdx$ and
$\displaystyle\int \frac{x}{\sqrt{1-x^4}}dx = \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}du$
$= \dfrac{1}{2}\sin^{-1}u + c = \dfrac{1}{2}\sin^{-1}x^2 + c$

(b) Let $u = 1 - x^4$ and then $du = -4x^3dx$ and
$\displaystyle\int \frac{x^3}{\left(1-x^4\right)^{1/2}}dx = -\frac{1}{4}\int u^{-1/2}du$
$= -\dfrac{1}{2}u^{1/2} + c = -\dfrac{1}{2}\left(1-x^4\right)^{1/2} + c$

24. (a) Let $u = x^3$ and then $du = 3x^2dx$ and
$\displaystyle\int \frac{x^2}{1+x^6}dx = \frac{1}{3}\int \frac{1}{1+u^2}du$
$= \dfrac{1}{3}\tan^{-1}u + c = \dfrac{1}{3}\tan^{-1}x^3 + c$

(b) Let $u = 1 + u^6$ and then $du = 6x^5dx$ and
$\displaystyle\int \frac{x^5}{1+x^6}dx = \frac{1}{6}\int \frac{1}{u}du$
$= \dfrac{1}{6}\ln|u| + c = \dfrac{1}{6}\ln\left|1+x^6\right| + c$

25. (a) $\displaystyle\int \frac{1+x}{1+x^2}dx$
$\displaystyle= \int \frac{1}{1+x^2}dx + \int \frac{x}{1+x^2}dx$
$\displaystyle= \tan^{-1}x + c_1 + \int \frac{x}{1+x^2}dx$
Let $u = 1 + x^2$ and then $du = 2xdx$.
$\displaystyle= \tan^{-1}x + c_1 + \frac{1}{2}\int \frac{1}{u}du$
$= \tan^{-1}x + c_1 + \dfrac{1}{2}\ln|u| + c_2$
$= \tan^{-1}x + \dfrac{1}{2}\ln\left|1+x^2\right| + c$
$= \tan^{-1}x + \dfrac{1}{2}\ln\left(1+x^2\right) + c$

(b) $\displaystyle\int \frac{1+x}{1-x^2}dx = \int \frac{1+x}{(1-x)(1+x)}dx$
$\displaystyle= \int \frac{1}{1-x}dx$
Let $u = 1 - x$ and then $du = -dx$.
$\displaystyle= -\int \frac{1}{u}du = -\ln|u| + c$
$= -\ln|1-x| + c$

26. (a) Let $u = x^{3/2}$ and then
$du = \frac{3}{2}x^{1/2}dx = \frac{3}{2}\sqrt{x}dx$ and
$\int \frac{3\sqrt{x}}{1+x^3}dx = 2\int \frac{1}{1+u^2}du$
$= 2\tan^{-1}u + c = 2\tan^{-1}\left(x^{3/2}\right) + c$

(b) Let $u = x^{\frac{5}{2}}$ and then
$du = \frac{5}{2}x^{\frac{3}{2}}dx = \frac{5}{2}x\sqrt{x}dx$ and
$\int \frac{x\sqrt{x}}{1+x^5}dx = \frac{2}{5}\int \frac{1}{1+u^2}du$
$= \frac{2}{5}\tan^{-1}u + c = \frac{2}{5}\tan^{-1}\left(x^{5/2}\right) + c$

27. Let $u = t+7$ and then $du = dt$, $t = u-7$ and
$\int \frac{2t+3}{t+7}dt = \int \frac{2(u-7)+3}{u}du$
$= \int \left(2 - \frac{11}{u}\right)du = 2u - 11\ln|u| + c$
$= 2(t+7) - 11\ln|t+7| + c$

28. Let $u = t+3$ and then $du = dt$ and
$\int \frac{t^2}{(t+3)^{1/3}}dt = \int \frac{(u-3)^2}{u^{1/3}}du$
$= \int \left(u^{5/3} - 6u^{2/3} + 9u^{-1/3}\right)du$
$= \frac{3}{8}u^{8/3} - \frac{18}{5}u^{5/3} + \frac{18}{2}u^{2/3} + c$
$= \frac{3}{8}(t+3)^{8/3} - \frac{18}{5}(t+3)^{5/3} + \frac{18}{2}(t+3)^{2/3} + c$

29. Let $u = \sqrt{1+\sqrt{x}}$ and then $(u^2-1)^2 = x$, $2(u^2-1)(2u)du = dx$ and
$\int \frac{1}{\sqrt{1+\sqrt{x}}}dx = \int \frac{4u(u^2-1)}{u}du$
$= 4\int (u^2-1)du = 4(\frac{u^3}{3} - u) + c$
$= \frac{4}{3}(1+\sqrt{x})^{3/2} - 4(1+\sqrt{x})^{1/2} + c$

30. Let $u = x^2$ and then $du = 2xdx$ and
$\int \frac{dx}{x\sqrt{x^4-1}} = \int \frac{du/2}{u\sqrt{u^2-1}}$
$= \frac{1}{2}\sec^{-1}u + c = \frac{1}{2}\sec^{-1}x^2 + c$

31. Let $u = x^2+1$ and then $u = 2xdx$, $u(0) = 1$, $u(2) = 5$ and
$\int_0^2 x\sqrt{x^2+1}dx = \frac{1}{2}\int_1^5 \sqrt{u}du$
$= \frac{1}{2}.\frac{2}{3}u^{3/2}\Big|_1^5 = \frac{1}{3}(\sqrt{125}-1) = \frac{5}{3}\sqrt{5} - \frac{1}{3}$

32. Let $u = \pi x^2$ and then $du = 2\pi xdx$ and
$\int_1^3 x\sin(\pi x^2)dx = \frac{1}{2\pi}\int_\pi^{9\pi} \sin udu = (\sin u)\Big|_\pi^{9\pi} = 0$

33. Let $u = t^2+1$ and then $du = 2tdt$,
$u(-1) = 2 = u(1)$ and
$\int_{-1}^1 \frac{t}{(t^2+1)^{1/2}}dt = \frac{1}{2}\int_2^2 u^{-1/2}du = 0$

34. Let $u = t^3$ and then $du = 3t^2dt$,
$u(0) = 0,\ u(2) = 8$ and
$\int_0^2 t^2e^{t^3}dt = \frac{1}{3}\int_0^8 e^u du = \frac{1}{3}e^u\Big|_0^8$
$= \frac{1}{3}\left(e^8 - 1\right)$

35. Let $u = e^x$ and then $du = e^x dx$,
$u(0) = 1$, $u(2) = e^2$ and
$\int_0^2 \frac{e^x}{1+e^{2x}}dx = \int_1^{e^2} \frac{1}{1+u^2}du = \tan^{-1}u\Big|_1^{e^2}$
$= \tan^{-1}e^2 - \frac{\pi}{4}$

36. Let $u = 1+e^x$ and then $du = e^x dx$,
$u(0) = 2$, $u(2) = 1+e^2$ and
$\int_0^2 \frac{e^x}{1+e^x}dx = \int_2^{1+e^2} \frac{1}{u}du = \ln(u)\Big|_2^{1+e^2}$
$= \ln\left(1+e^2\right) - \ln(2) = \ln\left(\frac{1+e^2}{2}\right)$

37. Let $u = \sin x$ and then $du = \cos xdx$
$u(\pi/4) = 1/\sqrt{2}, u(\pi/2) = 1$ and
$\int_{\pi/4}^{\pi/2} \cot xdx = \int_{1/\sqrt{2}}^1 \frac{1}{u}du = \ln|u|\Big|_{1/\sqrt{2}}^1$
$= \ln\sqrt{2}$

38. Let $u = \ln x$ and then $du = \frac{1}{x}dx$, $u(1) = 0$, $u(e) = 1$ and
$\int_1^e \frac{\ln x}{x}dx = \int_0^1 udu = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}$

39. $\int_1^4 \frac{x-1}{\sqrt{x}}dx = \int_1^4 (x^{1/2} - x^{-1/2})dx$
$= (\frac{2}{3}x^{3/2} - 2x^{1/2})\Big|_1^4$
$= (\frac{16}{3} - 4) - (\frac{2}{3} - 2) = \frac{8}{3}$

40. Let $u = x^2+1$ and then $du = 2xdx$ and
$\int_0^1 \frac{x}{(x^2+1)^{1/2}}dx = \frac{1}{2}\int_1^2 u^{-1/2}du$
$= (u^{1/2})\Big|_1^2 = \sqrt{2} - 1$

41. (a) $\int_0^\pi \sin x^2 dx \approx .77$ using midpoint evaluation with $n \geq 40$.

(b) Let $u = x^2$ and then $du = 2xdx$ and
$\int_0^{\pi} x \sin x^2 dx = \frac{1}{2}\int_0^{\pi^2} \sin u du$
$= \frac{1}{2}(-\cos u)\Big|_0^{\pi^2}$
$= -\frac{1}{2}\cos \pi^2 + \frac{1}{2}$
≈ 0.95134

42. (a) Let $u = x^2$ and then $du = 2xdx$, $u(-1) = 1$, $u(1) = 1$ and $\int_{-1}^{1} xe^{-x^2} dx = \frac{1}{2}\int_1^1 e^{-u} du = 0$

(b) $\int_{-1}^{1} e^{-x^2} \approx 1.4937$ using midpoint evaluation with $n \geq 50$.

43. (a) $\int_0^2 \frac{4x^2}{(x^2+1)^2} dx \approx 1.414$ using right endpoint evaluation with $n \geq 50$.

(b) Let $u = x^2 + 1$ and then $du = 2xdx$, $x^2 = u - 1$ and
$\int_0^2 \frac{4x^3}{(x^2+1)^2} dx = \int_1^5 2.\frac{u-1}{u^2} du$
$= \int_1^5 (2u^{-1} - 2u^{-2}) du$
$= (2\ln|u| + 2u^{-1})\big|_1^5 = 2\ln 5 - \frac{8}{5}$

44. (a) $\int_0^{\pi/4} \sec x dx \approx .88$ using midpoint evaluation with $n \geq 10$.

(b) $\int_0^{\pi/4} \sec^2 x dx = (\tan x)\Big|_0^{\pi/4} = 1.$

45. $\frac{1}{2}\int_0^4 f(u)du.$

46. $\frac{1}{3}\int_1^8 f(u)du.$

47. $\int_0^1 f(u)du.$

48. $\int_0^4 \frac{f(\sqrt{x})}{\sqrt{x}} dx = 2\int_0^2 f(u)du.$

49. $\int_{-a}^{a} f(x)dx$
$= \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx$
Let $u = -x$ and $du = -dx$ in the first integral. Then
$\int_{-a}^{a} f(x)$
$= -\int_{-a}^{0} f(-u)du + \int_0^a f(x)dx$
$= \int_0^a f(-u)du + \int_0^a f(x)dx$
If f is even, then $f(-u) = f(u)$, and so
$\int_{-a}^{a} f(x)dx$
$= \int_0^a f(u)du + \int_0^a f(x)dx$
$= \int_0^a f(x)dx + \int_0^a f(x)dx$
$= 2\int_0^a f(x)dx$
If f is odd, then $f(-u) = -f(u)$, and so
$\int_{-a}^{a} f(x)dx$
$= -\int_0^a f(u)du + \int_0^a f(x)dx$
$= -\int_0^a f(x)dx + \int_0^a f(x)dx$
$= 0$

50. First, let $u = x - T$, then for any a,
$\int_T^{a+T} f(x)dx = \int_0^a f(u+T)du$
$= \int_0^a f(u)du = \int_0^a f(x)dx$
If we let $a = T$, then we get
$\int_a^T f(x)dx = \int_T^{2T} f(x)dx.$
If we let $a = 2T$, then we get
$\int_0^{2T} f(x)dx = \int_T^{3T} f(x)dx$
and then
$\int_0^T f(x)dx = \int_T^{2T} f(x)dx$
$= \int_0^{2T} f(x)dx - \int_0^T f(x)dx$
$= \int_T^{3T} f(x)dx - \int_T^{2T} f(x)dx$
$= \int_{2T}^{3T} f(x)dx$
It is straight forward to see that for any integer i,
$$\int_0^T f(x)dx = \int_{iT}^{(i+1)T} f(x)dx$$
Now suppose $0 \leq a \leq T$, then
$$\int_0^T f(x)dx - \int_a^{a+T} dx$$

$= \int_0^a f(x)dx - \int_T^{a+T} f(x)dx$

So $\int_0^T f(x)dx = \int_a^{a+T} dx$

Now suppose a is any number. Then a must lie in some interval $[iT, (i+1)T]$ for some interger i. Use the similar method as in above, we shall get

$\int_{iT}^{(i+1)T} f(x)dx = \int_a^{a+T} f(x)dx$

And since $\int_{iT}^{(i+1)T} f(x)dx = \int_0^T f(x)dx$,

we get $\int_0^T f(x)dx = \int_a^{a+T} f(x)dx$

51. (a) Let $u = 10 - x$, so that $du = -dx$. Then,

$$I = \int_0^{10} \frac{\sqrt{x}}{\sqrt{x} + \sqrt{10-x}} dx$$
$$= -\int_{x=0}^{x=10} \frac{\sqrt{10-u}}{\sqrt{10-u} + \sqrt{u}} du$$
$$= -\int_{u=10}^{u=0} \frac{\sqrt{10-u}}{\sqrt{10-u} + \sqrt{u}} du$$
$$= \int_{u=0}^{u=10} \frac{\sqrt{10-u}}{\sqrt{10-u} + \sqrt{u}} du$$
$$I = \int_{x=0}^{x=10} \frac{\sqrt{10-x}}{\sqrt{10-x} + \sqrt{x}} dx$$

The last equation follows from the previous one because u and x are dummy variables of integration. Now note that

$$\frac{\sqrt{x}}{\sqrt{x} + \sqrt{10-x}}$$
$$= \frac{\sqrt{x} + \sqrt{10-x} - \sqrt{10-x}}{\sqrt{x} + \sqrt{10-x}}$$
$$= 1 - \frac{\sqrt{10-x}}{\sqrt{x} + \sqrt{10-x}}$$

Thus,

$$\int_0^{10} \frac{\sqrt{x}}{\sqrt{x} + \sqrt{10-x}} dx$$
$$= \int_0^{10} \left[1 - \frac{\sqrt{10-x}}{\sqrt{x} + \sqrt{10-x}}\right] dx$$
$$= \int_0^{10} 1dx - \int_0^{10} \frac{\sqrt{10-x}}{\sqrt{x} + \sqrt{10-x}} dx$$
$$I = \int_0^{10} 1dx - I$$
$$2I = 10$$
$$I = 5$$

(b) Let $u = a - x$, so that $du = -dx$ Then,

$$I = \int_0^a \frac{f(x)}{f(x) + f(a-x)} dx$$
$$= -\int_a^0 \frac{f(a-u)}{f(a-u) + f(u)} du$$
$$= \int_0^a \frac{f(a-u)}{f(a-u) + f(u)} du$$
$$I = \int_0^a \frac{f(a-x)}{f(a-x) + f(x)} dx$$

The last equation follows from the previous one because u and x are dummy variables of integration. Now note that

$$\frac{f(x)}{f(x) + f(a-x)}$$
$$= \frac{f(x) + f(a-x) - f(a-x)}{f(x) + f(a-x)}$$
$$= 1 - \frac{f(a-x)}{f(a-x) + f(x)}$$

Thus,

$$\int_0^a \frac{f(x)}{f(x) + f(a-x)} dx$$
$$= \int_0^a \left[1 - \frac{f(a-x)}{f(a-x) + f(x)}\right] dx$$
$$= \int_0^a 1dx - \int_0^a \frac{f(a-x)}{f(a-x) + f(x)} dx$$
$$2I = a$$
$$I = a/2$$

52. (a) Let $u = 6 - x$, so that $du = -dx$. Then,

$$I = \int_2^4 \frac{\sin^2(9-x)}{\sin^2(9-x) + \sin^2(x+3)} dx$$
$$= -\int_4^2 \frac{\sin^2(u+3)}{\sin^2(u+3) + \sin^2(9-u)} du$$
$$= \int_2^4 \frac{\sin^2(u+3)}{\sin^2(u+3) + \sin^2(9-u)} du$$
$$= \int_2^4 \frac{\sin^2(x+3)}{\sin^2(x+3) + \sin^2(9-x)} dx$$
$$= \int_2^4 \left[1 - \frac{\sin^2(9-x)}{\sin^2(x+3) + \sin^2(9-x)}\right] dx$$
$$I = \int_2^4 1dx - I$$
$$2I = 2$$
$$I = 1$$

(b) Let $u = 6 - x$, so that $du = -dx$. Then,

$$I = \int_2^4 \frac{f(9-x)}{f(9-x) + f(x+3)} dx$$
$$= -\int_4^2 \frac{f(u+3)}{f(u+3) + f(9-u)} du$$
$$= \int_2^4 \frac{f(u+3)}{f(u+3) + f(9-u)} du$$
$$= \int_2^4 \frac{f(x+3)}{f(x+3) + f(9-x)} dx$$

$$= \int_2^4 \left[1 - \frac{f(9-x)}{f(x+3)+f(9-x)}\right] dx$$
$$I = \int_2^4 1dx - I$$
$$2I = 2$$
$$I = 1$$

53. Let $6-u = x+4$; that is, let $u = 2-x$, so that $du = -dx$.
Then,
$$I = \int_0^2 \frac{f(x+4)}{f(x+4)+f(6-x)}dx$$
$$= -\int_2^0 \frac{f(6-u)}{f(6-u)+f(u+4)}du$$
$$= \int_0^2 \frac{f(6-u)}{f(6-u)+f(u+4)}du$$
$$= \int_0^2 \frac{f(6-x)}{f(6-x)+f(x+4)}dx$$
$$= \int_0^2 \frac{f(6-x)+f(x+4)-f(x+4)}{f(6-x)+f(x+4)}dx$$
$$= \int_0^2 \left[1 - \frac{f(x+4)}{f(6-x)+f(x+4)}\right] dx$$
$$I = \int_0^2 1dx - I$$
$$2I = 2$$
$$I = 1$$

54. (a) Let $u = x^{1/6}$, so that $du = \frac{1}{6}x^{-5/6}dx$.
Then,
$$I = \int \frac{1}{x^{5/6}+x^{2/3}}dx$$
$$= \int \frac{x^{-5/6}dx}{1+x^{-1/6}}$$
$$= \int \frac{6\,du}{1+\frac{1}{u}}$$
$$= \int \frac{6u}{u+1}\,du$$
Let $v = u+1$, then $dv = du$ and $u = v-1$.
Then, $I = \displaystyle\int \frac{6u}{u+1}\,du$
$$= \int \frac{6(v-1)}{v}\,dv$$
$$= \int \left(6 - \frac{6}{v}\right) dv$$
$$= 6v - 6\ln|v| + c$$
$$= 6(u+1) - 6\ln|u+1| + c$$
$$= 6(x^{1/6}+1) - 6\ln|x^{1/6}+1| + c$$

(b) Let $u = x^{1/6}$, so that $du = (1/6)x^{-5/6}dx$, which means $6u^5du = dx$.
Thus,
$$\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}dx$$
$$= 6\int \frac{u^5}{u^3+u^2}du$$
$$= 6\int \frac{u^3}{u+1}du$$
$$= 6\int \left[u^2 - u + 1 - \frac{1}{u+1}\right] du$$
$$= 6\left[\frac{u^3}{3} - \frac{u^2}{2} + u - \ln|u+1|\right] + c$$
$$= 2x^{1/2} - 3x^{1/3} + 6x^{1/6}$$
$$= -6\ln|x^{1/6}+1| + c$$

(c) Let $u = x^{1/q}$, then $q\,du = x^{(1-q)/q}dx$, and
$$I = \int \frac{1}{x^{(p+1)/q}+x^{p/q}}dx$$
$$= \int \frac{x^{(1-q)/q}dx}{x^{(p+2-q)/q}+x^{(p+1-q)/q}}dx$$
$$= q\int \frac{1}{u^{p+2-q}+u^{p+1-q}}\,du$$
$$= q\int \frac{u^{q-1-p}}{u+1}du$$
The rest of the calculation will depend on the values of p and q.

55. First let $u = \ln\sqrt{x}$, so that $du = x^{-1/2}(1/2)x^{-1/2}dx$, so that $2du = \frac{1}{x}dx$. Then,
$$\int \frac{1}{x\ln\sqrt{x}}dx = 2\int \frac{1}{u}du$$
$$= 2\ln|u| + c$$
$$= 2\ln|\ln\sqrt{x}| + c$$

Now use the substitution $u = \ln x$, so that $du = \frac{1}{x}dx$. Then,
$$\int \frac{1}{x\ln\sqrt{x}}dx = \int \frac{1}{x\ln(x^{1/2})}dx$$
$$= \int \frac{1}{x\left(\frac{1}{2}\right)\ln x}dx$$
$$= 2\int \frac{1}{u}du$$
$$= 2\ln|u| + c_1$$
$$= 2\ln|\ln x| + c_1$$

The two results differ by a constant, and so are equivalent, as can be seen as follows:
$2\ln|\ln\sqrt{x}| = 2\ln|\ln(x^{1/2})|$
$$= 2\ln\left|\frac{1}{2}\ln x\right|$$
$$= 2\left[\ln\frac{1}{2} + \ln|\ln x|\right]$$
$$= 2\ln\frac{1}{2} + 2\ln|\ln x|$$
$$= 2\ln|\ln x| + \text{ constant}$$

56. The area of the region bounded by the curve $y = \pi x - x^2$ and x-axis, where $0 \le x \le 1$ is
$$\int_0^1 \left(\pi x - x^2\right) dx$$

$= \left(\pi \frac{x^2}{2} - \frac{x^3}{3} \right) \Big|_0^1$
$= \frac{\pi}{2} - \frac{1}{3}.$

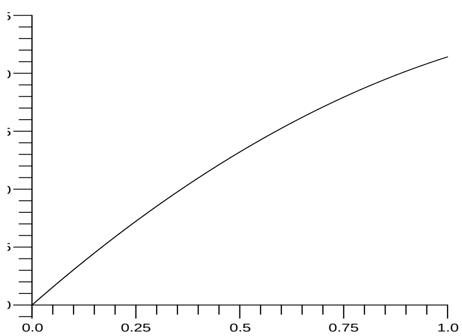
The area of the region bounded by the curve $y = \left(\pi \cos x - \cos^2 x\right) \sin x$ and x-axis, where $0 \le x \le \frac{\pi}{2}$ is $\int_0^{\pi/2} \left(\pi \cos x - \cos^2 x\right) \sin x dx$.

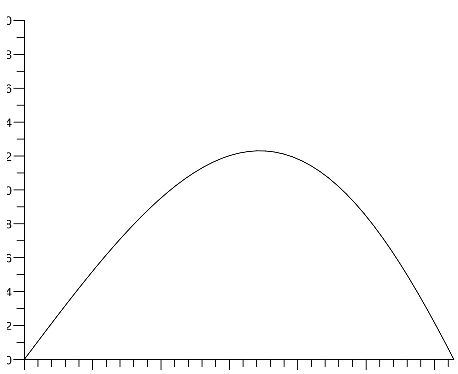
Let $u = \cos x$ and then $du = -\sin x dx$.

$u\left(0\right) = 1, u\left(\frac{\pi}{2}\right) = 0.$

$= \int_1^0 \left(-\pi u + u^2\right) du$
$= -\pi \left(\frac{u^2}{2}\right) + \frac{u^3}{3} \Big|_1^0$
$= \frac{\pi}{2} - \frac{1}{3}$

Thus, the areas are equal.





57. The point is that if we let $u = x^4$, then we get $x = \pm u^{1/4}$, and so we need to pay attention to the sign of u and x. A safe way is to solve the original indefinite integral in terms of x, and then solve the definite integral using boundary points in terms of x.

$\int_{-2}^{1} 4x^4 dx = \int_{x=-2}^{x=1} u^{1/4} du$
$= \frac{4}{5} u^{5/4} \Big|_{x=-2}^{x=1} = \frac{4}{5} x^5 \Big|_{x=-2}^{x=1}$
$= \frac{4}{5}\left(1^5 - (-2)^5\right) = \frac{4}{5}(1 - (-32)) = \frac{132}{5}$

58. The problem is that it is not true on entire interval $[0, \pi]$ that $\cos x = \sqrt{1 - \sin^2 x}$. This is only true on the interval $\left[0, \frac{\pi}{2}\right]$. To make this substitution correctly, one must break up the integral:

$\int_0^{\pi} \cos x (\cos x) dx$
$= \int_0^{\pi/2} \cos x (\cos x) dx + \int_{\pi/2}^{\pi} \cos x (\cos x) dx$
$= \int_{x=0}^{x=\pi/2} \sqrt{1 - u^2} du$
$- \int_{x=\pi/2}^{x=\pi} \sqrt{1 - u^2} du$
$= \left(\frac{u}{2} + \frac{\sin^{-1} u}{2} \right) \Big|_{x=0}^{x=\pi/2}$
$- \left(\frac{u}{2} + \frac{\sin^{-1} u}{2} \right) \Big|_{x=\pi/2}^{x=\pi}$
$= \left(\frac{\sin x}{2} + \frac{\sin^{-1}(\sin x)}{2} \right) \Big|_{x=0}^{x=\pi/2}$
$- \left(\frac{\sin x}{2} + \frac{\sin^{-1}(\sin x)}{2} \right) \Big|_{x=\pi/2}^{x=\pi}$
$= \left(\frac{1}{2} + \frac{\pi}{4} \right) - 0 - 0 + \left(\frac{1}{2} + \frac{\pi}{4} \right)$
$= 1 + \frac{\pi}{2}$

59. Let $u = 1/x$, so that $du = -1/x^2 dx$, which means that $-1/u^2 du = dx$. Then,

$\int_0^1 \frac{1}{x^2 + 1} dx = -\int_{1/a}^{1} \frac{1/u^2}{1/u^2 + 1} du$
$= \int_1^{1/a} \frac{1}{1 + u^2} du = \int_1^{1/a} \frac{1}{1 + x^2} dx$

The last equation follows from the previous one because u and x are dummy variables of integration. Thus,

$\tan^{-1} x \Big|_a^1 = \tan^{-1} x \Big|_1^{1/a}$
$\tan^{-1} 1 - \tan^{-1} a = \tan^{-1} \frac{1}{a} - \tan^{-1} 1$
$\tan^{-1} a + \tan^{-1} \frac{1}{a} = 2 \tan^{-1} 1$
$\tan^{-1} a + \tan^{-1} \frac{1}{a} = \frac{\pi}{2}$

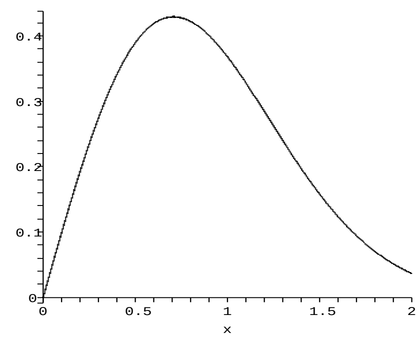
60. If $u = 1/x$, then $du = -dx/x^2$ and

$\int \frac{1}{|x| \sqrt{x^2 - 1}} dx$
$= \int \frac{1}{x^2 \sqrt{x^2 - 1}} dx$
$= -\int \frac{1}{\sqrt{1 - u^2}} du$
$= -\sin^{-1} u + c$
$= -\sin^{-1} 1/x + c$

On the other hand,
$\int \frac{1}{|x|\sqrt{x^2-1}}dx = \sec^{-1}x + c_1$
So $-\sin^{-1}1/x = \sec^{-1}x + c_2$.
Let $x = 1$, we get
$\sin^{-1}1 = \sec^{-1}1 + c_2$
$\frac{\pi}{2} = 0 + c_2$
$c_2 = \frac{\pi}{2}$

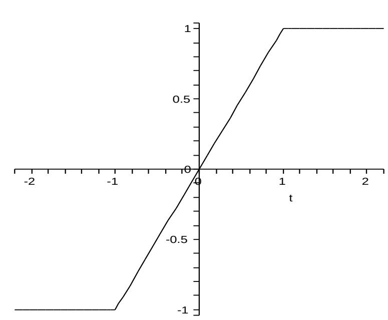
61. $\overline{x} = \frac{\int_{-2}^{2} x\sqrt{4-x^2}dx}{\int_{-2}^{2}\sqrt{4-x^2}dx}$
Examine the denominator of $\overline{x}$, the graph of $\sqrt{4-x^2}$, which is indeed a semicircle, is symmetric over the two intervals $[-2,0]$ and $[0,2]$, while multiplying by x changes the symmetry into anti-symmetry. In other words,
$\int_{-2}^{0} x\sqrt{4-x^2}dx = -\int_{0}^{2} x\sqrt{4-x^2}dx$
so that
$\int_{-2}^{2} x\sqrt{4-x^2}dx$
$= \int_{-2}^{0} x\sqrt{4-x^2}dx + \int_{0}^{2} x\sqrt{4-x^2}dx = 0$
Hence $\overline{x} = 0$.
Now the integral $\int_{-2}^{2}\sqrt{4-x^2}dx$ is the area of a semicircle with radius 2, thus its value is $(1/2)\,\pi 2^2 = 2\pi$. Then
$\overline{y} = \frac{\int_{-2}^{2}\left(\sqrt{4-x^2}\right)^2dx}{2\int_{-2}^{2}\sqrt{4-x^2}dx}$
$= \frac{\int_{-2}^{2}\left(4-x^2\right)dx}{2.2\pi}$
$= \frac{\int_{-2}^{0}\left(4-x^2\right)dx + \int_{0}^{2}\left(4-x^2\right)dx}{4\pi}$
$= \frac{2\int_{0}^{2}\left(4-x^2\right)dx}{4\pi}$
$= \frac{\int_{0}^{2}\left(4-x^2\right)dx}{2\pi}$
$= \frac{1}{2\pi}\left(4x - \frac{x^3}{3}\right)\Big|_0^2 = \frac{8}{3\pi}$

62. These animals are likely to be found 0.7 miles from the pond. Let $u = -x^2$, then $du = -2xdx$, $u(0) = 0$, $u(2) = -4$ and
$\int_0^2 xe^{-x^2}dx = -\frac{1}{2}\int_0^{-4} e^u du$
$= -\frac{1}{2}\left(e^{-4}-1\right) = \frac{1-e^{-4}}{2}$



63. $V(t) = V_p \sin(2\pi f t) V^2(t)$
$= V_p^2 \sin^2(2\pi f t)$
$= V_p^2\left(\frac{1}{2} - \frac{1}{2}\cos(4\pi f t)\right)$
$= \frac{V_p^2}{2}\left(1-\cos(4\pi f t)\right)$
$\text{rms} = \sqrt{f\int_0^{1/f} V^2(t)\,dt}$
$= \sqrt{f\int_0^{1/f}\frac{V_p^2}{2}\left(1-\cos(4\pi f t)\right)dt}$
$= \frac{V_p\sqrt{f}}{\sqrt{2}}\sqrt{\left(t - \frac{\sin(4\pi f t)}{4\pi f}\right)\Big|_0^{1/f}}$
$= \frac{V_p\sqrt{f}}{\sqrt{2}}\sqrt{\frac{1}{f}} = \frac{V_p}{\sqrt{2}}$

64. $\int_{-2}^{2} f^2(t)dt$
$= \int_{-2}^{-1} 1dt + \int_{-1}^{1} t^2dt + \int_{1}^{2} 1dt$
$= 1 + \frac{2}{3} + 1 = \frac{8}{3}$
$\text{rms} = \sqrt{\frac{1}{4}\int_1^2 f^2(t)\,dt}$
$= \sqrt{\frac{1}{4}\left(\frac{8}{3}\right)} = \sqrt{\frac{2}{3}}$



4.7 Numerical Integration

1. Midpoint Rule:

$\int_0^1 (x^2+1)\,dx$
$\approx \frac{1}{4}\left[f\left(\frac{1}{8}\right)+f\left(\frac{3}{8}\right)+f\left(\frac{5}{8}\right)+f\left(\frac{7}{8}\right)\right]$
$=\frac{85}{64}$

Trapezoidal Rule:

$\int_0^1 (x^2+1)\,dx$
$\approx \frac{1-0}{2(4)}\left[f(0)+2f\left(\frac{1}{4}\right)+2f\left(\frac{1}{2}\right)+2f\left(\frac{3}{4}\right)\right.$
$\left.+f(1)\right]$
$=\frac{43}{32}$

Simpson's Rule:

$\int_0^1 (x^2+1)\,dx$
$=\frac{1-0}{3(4)}\left[f(0)+4f\left(\frac{1}{4}\right)+2f\left(\frac{1}{2}\right)+4f\left(\frac{3}{4}\right)\right.$
$\left.+f(1)\right]$
$=\frac{4}{3}$

2. Midpoint Rule:

$\int_0^2 (x^2+1)\,dx$
$\approx \frac{1}{2}\left[f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)+f\left(\frac{5}{4}\right)+f\left(\frac{7}{4}\right)\right]$
$=\frac{1}{2}\left(\frac{17}{16}+\frac{25}{16}+\frac{41}{16}+\frac{65}{16}\right)$
$=\frac{37}{8}$

Trapezoidal Rule:

$\int_0^2 (x^2+1)\,dx$
$\approx \frac{1}{4}\left[f(0)+2f\left(\frac{1}{2}\right)+2f(1)+2f\left(\frac{3}{2}\right)\right.$
$\left.+f(2)\right]$
$=\frac{1}{4}\left(1+\frac{5}{2}+4+\frac{13}{2}+5\right)$
$=\frac{19}{4}$

Simpson's Rule:

$\int_0^2 (x^2+1)\,dx$
$=\frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+2f(1)+4f\left(\frac{3}{2}\right)\right.$
$\left.+f(2)\right]$
$=\frac{1}{6}(1+5+4+13+5)$
$=\frac{14}{3}$

3. Midpoint Rule:

$\int_1^3 \frac{1}{x}dx$
$\approx \frac{3-1}{4}\left[f\left(\frac{5}{4}\right)+f\left(\frac{7}{4}\right)+f\left(\frac{9}{4}\right)+f\left(\frac{11}{4}\right)\right]$
$=\frac{1}{2}\left(\frac{4}{5}+\frac{4}{7}+\frac{4}{9}+\frac{4}{11}\right)$
$=\frac{3776}{3465}$

Trapezoidal Rule:

$\int_1^3 \frac{1}{x}dx$
$\approx \frac{3-1}{2(4)}\left[f(1)+2f\left(\frac{3}{2}\right)+2f(2)+2f\left(\frac{5}{2}\right)\right.$
$\left.+f(3)\right]$
$=\frac{1}{4}\left(1+\frac{4}{3}+1+\frac{4}{5}+\frac{1}{3}\right)$
$=\frac{67}{60}$

Simpson's Rule:

$\int_1^3 \frac{1}{x}dx$
$=\frac{3-1}{3(4)}\left[f(1)+4f\left(\frac{3}{2}\right)+2f(2)+4f\left(\frac{5}{2}\right)\right.$
$\left.+f(3)\right]$
$=\frac{1}{6}\left(1+\frac{8}{3}+1+\frac{8}{5}+\frac{1}{3}\right)$
$=\frac{11}{10}$

4. Midpoint Rule:

$\int_{-1}^1 (2x-x^2)\,dx$
$\approx \frac{1}{2}\left[f\left(-\frac{3}{4}\right)+f\left(-\frac{1}{4}\right)+f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)\right]$
$=\frac{1}{2}\left(-\frac{33}{16}-\frac{9}{16}+\frac{7}{16}+\frac{15}{16}\right)$
$=\frac{-5}{8}$

Trapezoidal Rule:

$\int_{-1}^1 (2x-x^2)\,dx$
$\approx \frac{1}{4}\left[f(-1)+2f\left(-\frac{1}{2}\right)+2f(0)+2f\left(\frac{1}{2}\right)\right.$
$\left.+f(1)\right]$

$= \frac{1}{4}\left(-3 - \frac{5}{2} + 0 + \frac{3}{2} + 1\right)$
$= -\frac{3}{4}$

Simpson's Rule:

$\int\limits_{-1}^{1} \left(2x - x^2\right) dx$
$\approx \frac{1}{6}\left[f(-1) + 4f\left(-\frac{1}{2}\right) + 2f(0) + 4f\left(\frac{1}{2}\right)\right.$
$\left.+f(1)\right]$
$= \frac{1}{6}(-3 - 5 + 0 + 3 + 1)$
$= -\frac{2}{3}$

5. Midpoint Rule:

$\ln 4 - 1.366162 = 1.386294 - 1.366162$
$= 0.020132$

Trapezoidal Rule:

$\ln 4 - 1.428091 = 1.386294 - 1.428091$
$= -0.041797$

Simpson's Rule:

$\ln 4 - 1.391621 = 1.386294 - 1.391621$
$= -0.005327$

Hence, the approximation using Simpson's Rule is too small and the Approximation using Trapezoidal Rule is too large.

6. Midpoint Rule:
$\ln 8 - 1.987287 = 2.079442 - 1.987287$
$= 0.092155$
Trapezoidal Rule:
$\ln 8 - 2.289628 = 2.079442 - 2.289628$
$= -0.210186$
Simpson's Rule:
$\ln 8 - 2.137327 = 2.079442 - 2.137327$
$= -0.057885$
Hence, the approximation using Simpson's Rule is too small and the Approximation using Trapezoidal Rule is too large.

7. Midpoint Rule:
$\sin 1 - 0.843666 = 0.841471 - 0.843666$
$= -0.002195$
Trapezoidal Rule: $\sin 1 - 0.837084 = 0.841471 - 0.837084$
$= 0.004387$
Simpson's Rule:
$\sin 1 - 0.841489 = 0.841471 - 0.841489$
$= -0.000018$
Hence, the approximation using Simpson's Rule is too small and the Approximation using Trapezoidal Rule is too large.

8. Midpoint Rule: $e^2 - 7.322986 = 7.389056 - 7.322986$
$= 0.06607$
Trapezoidal Rule: $e^2 - 7.52161 = 7.389056 - 7.52161$
$= -0.132554$
Simpson's Rule: $e^2 - 7.391210 = 7.389056 - 7.391210$
$= -0.002154$
Hence, the approximation using Simpson's Rule is too small and the Approximation using Trapezoidal Rule is too large.

9. $\int\limits_0^{\pi} \cos x^2 dx$

n	*Midpoint*	*Trapezoidal*	*Simpson*
10	0.5538	0.5889	0.5660
20	0.5629	0.5713	0.5655
50	0.5652	0.566	0.5657

10. $\int\limits_0^{\frac{\pi}{4}} \sin \pi x^2 dx$

n	*Midpoint*	*Trapezoidal*	*Simpson*
10	0.386939	0.385578	0.386476
20	0.386600	0.386259	0.386485
50	0.386504	0.386450	0.386486

11. $\int\limits_0^{2} e^{-x^2} dx$

n	*Midpoint*	*Trapezoidal*	*Simpson*
10	0.88220	0.88184	0.88207
20	0.88211	0.88202	0.88208
50	0.88209	0.88207	0.88208

12. $\int\limits_0^{3} e^{-x^2} dx$

n	*Midpoint*	*Trapezoidal*	*Simpson*
10	0.886210	0.886202	0.886207
20	0.886208	0.886206	0.886207
50	0.886207	0.886207	0.886207

13. $\int\limits_0^{\pi} e^{\cos x} dx$

n	*Midpoint*	*Trapezoidal*	*Simpson*
10	3.9775	3.9775	3.9775
20	3.9775	3.9775	3.9775
50	3.9775	3.9775	3.9775

14. $\int\limits_0^{1} \sqrt[3]{x^2 + 1} dx$

n	*Midpoint*	*Trapezoidal*	*Simpson*
10	3.333017	3.336997	3.334337
20	3.334012	3.335007	3.334344
50	3.334291	3.334450	3.334344

15. The exact value of this integral is
$\int_0^1 5x^4 dx = x^5\big|_0^1 = 1 - 0 = 1$

n	*Midpoint*	EM_n
10	1.00832	8.3×10^{-3}
20	1.00208	2.1×10^{-3}
40	1.00052	5.2×10^{-3}
80	1.00013	1.3×10^{-3}

n	*Trapezoidal*	ET_n
10	0.98335	1.6×10^{-2}
20	0.99583	4.1×10^{-3}
40	0.99869	1.0×10^{-3}
80	0.99974	2.6×10^{-4}

n	*Simpson*	ES_n
10	1.000066	6.6×10^{-5}
20	1.0000041	4.2×10^{-6}
40	1.00000026	2.6×10^{-7}
80	1.00000016	1.6×10^{-8}

16. The exact value of this integral is
$\int_1^2 \frac{1}{x} dx = \ln 2$

n	*Midpoint*	EM_n
10	0.692835	3.1×10^{-4}
20	0.693069	7.8×10^{-5}
40	0.693128	2.0×10^{-5}
80	0.693142	4.9×10^{-6}

n	*Trapezoidal*	ET_n
10	0.693771	6.2×10^{-4}
20	0.693303	1.6×10^{-4}
40	0.693186	3.9×10^{-5}
80	0.693157	9.8×10^{-6}

n	Simpson	ES_n
10	0.693150	3.1×10^{-6}
20	0.693147	1.9×10^{-7}
40	0.693147	1.2×10^{-8}
80	0.693147	8.0×10^{-10}

17. The exact value of this integral is
$\int_0^\pi \cos x dx = \sin x|_0^\pi = 0$

n	*Midpoint*	EM_n
10	0	0
20	0	0
40	0	0
80	0	0

n	*Trapezoidal*	ET_n
10	0	0
20	0	0
40	0	0
80	0	0

n	*Simpson*	ES_n
10	0	0
20	0	0
40	0	0
80	0	0

18. The exact value of this integral is
$\int_0^{\frac{\pi}{4}} \cos x dx = \frac{1}{\sqrt{2}}$

n	*Midpoint*	EM_n
10	0.707289	1.8×10^{-4}
20	0.707152	4.5×10^{-5}
40	0.707118	1.1×10^{-5}
80	0.707110	2.8×10^{-6}

n	*Trapezoidal*	ET_n
10	0.706743	3.6×10^{-4}
20	0.707016	9.1×10^{-5}
40	0.707084	2.3×10^{-5}
80	0.707101	5.7×10^{-6}

n	*Simpson*	ES_n
10	0.7071087	1.5×10^{-7}
20	0.7071068	9.5×10^{-9}
40	0.7071068	6×10^{-10}
80	0.7071068	6×10^{-10}

19. If you double the error in the Midpoint Rule is divided by 4, the error in the Trapezoidal Rule is divided by 4 and the error in the Simpson's Rule is divided by 16.

20. If you halve the interval length $b - a$ the error in the Midpoint Rule is divided by 8, the error in the Trapezoidal Rule is divided by 8 and the error in the Simpson's Rule is divided by 32.

21. Trapezoidal Rule:
$\int_0^2 f(x)\,dx$
$\approx \frac{2-0}{2(8)}\left[f(0) + 2f(0.25) + 2f(0.5)\right.$
$+ 2f(0.75) + 2f(1) + 2f(1.25) + 2f(1.5)$
$\left.+ 2f(1.75) + f(2)\right]$
$= \frac{1}{8}[4.0 + 9.2 + 10.4 + 9.6 + 10 + 9.2 + 8.8$
$+ 7.6 + 4.0]$
$= 9.1$
Simpson's Rule:
$\int_0^2 f(x)\,dx$
$\approx \frac{2-0}{3(8)}\left[f(0) + 4f(0.25) + 2f(0.5)\right.$
$+ 4f(0.75) + 2f(1) + 4f(1.25) + 2f(1.5)$
$\left.+ 4f(1.75) + f(2)\right]$

$= \frac{1}{12}[4.0 + 18.4 + 10.4 + 19.2 + 10.0$
$+18.4 + 8.8 + 15.2 + 4.0]$
≈ 9.033

22. Trapezoidal Rule:
$\int_0^2 f(x)\,dx$
$\approx \frac{0.25}{2}[f(0) + 2f(0.25) + 2f(0.5)$
$+2f(0.75) + 2f(1) + 2f(1.25) + 2f(1.5)$
$+2f(1.75) + f(2)]$
$= \frac{0.25}{2}[(1.0) + 2(0.6) + 2(0.2) + 2(-0.2)$
$+ 2(-0.4) + 2(0.4) + 2(0.8)$
$+ 2(1.2) + (2.0)]$
$= 1.025.$
Simpson's Rule:
$\int_0^2 f(x)\,dx$
$\approx \frac{0.25}{3}[f(0) + 4f(0.25) + 2f(0.5)$
$+ 4f(0.75) + 2f(1) + 4f(1.25) + 2f(1.5)$
$+4f(1.75) + f(2)]$
$= \frac{0.25}{3}[(1.0) + 4(0.6) + 2(0.2) + 4(-0.2)$
$+2(-0.4) + 4(0.4) + 2(0.8) + 4(1.2) + (2.0)]$
≈ 1.016667

23. (a) $f(x) = \frac{1}{x}, f''(x) = \frac{2}{x^3}, f^{(4)}(x) = \frac{24}{x^5}$. Then $K = 2$, $L = 24$. Hence according to Theorems 9.1 and 9.2,
$|ET_4| \le 2\frac{(4-1)^3}{12 \cdot 4^2} \approx 0.281$
$|EM_4| \le 2\frac{(4-1)^3}{24 \cdot 4^2} \approx 0.141$
$|ES_4| \le 24\frac{(4-1)^5}{180 \cdot 4^2} \approx 0.127$

(b) Using Theorems 9.1 and 9.2, and the calculation in Example 9.10, we find the following lower bounds for the number of steps needed to guarantee accuracy of 10^{-7} in Exercise 5:
Midpoint: $\sqrt{\frac{2 \cdot 3^3}{24 \cdot 10^{-7}}} \approx 4745$
Trapezoidal: $\sqrt{\frac{2 \cdot 3^3}{14 \cdot 10^{-7}}} \approx 6709$
Simpson's: $\sqrt[4]{\frac{24 \cdot 3^5}{180 \cdot 10^{-7}}} \approx 135$

24. (a) $f(x) = \cos x, f''(x) = -\cos x,$
$f^{(4)}(x) = \cos x$.Then $K = L = 1$.
Hence according to
Theorems 9.1 and 9.2,
$|ET_4| \le 1\frac{1}{12 \cdot 4^2} \approx 0.005$
$|EM_4| \le 1\frac{1}{24 \cdot 4^2} \approx 0.003$
$|ES_4| \le 1\frac{1}{180 \cdot 4^4} \approx 2.17 \times 10^{-5}$

(b) Midpoint: $|E_n|K\frac{(b-a)^3}{24n^2} = \frac{1}{24n^2}$

We want $\frac{1}{24n^2} \le 10^7$
$24n^2 \ge 10^7$
$n^2 \ge \frac{10^7}{24}$
$n \ge \sqrt{\frac{10^7}{24}} \approx 645.5$
So need $n \ge 646$.
Trapezoid: $|ET_n|K\frac{(b-a)^3}{12n^2} = \frac{1}{12n^2}$
We want $n^2 \ge \frac{10^7}{12}$
$n \ge \sqrt{\frac{10^7}{12}} \approx 912.87$
$n \ge 913$
Simpson: $|ES_n|L\frac{(b-a)^5}{180n^4} = \frac{1}{180n^4}$
$\frac{1}{180n^4} \le 10^{-7}$
$180n^4 \ge 10^7$
$n^4 \ge \frac{10^7}{180}$
$n \ge \sqrt[4]{\frac{10^7}{180}} \approx 15.4$
So need $n \ge 16$.

25. (a) $f(x) = \ln x$. Hence, $f'(x) = \frac{1}{x}$ and $f''(x) = -\frac{1}{x^2}$. Therefore $|f''(x)| \le 1$.
The error using Trapezoidal Rule is
$|E(T_n)| \le 1\frac{(2-1)^3}{12n^2} \le 10^{-6}$
$|E(T_n)| \le \frac{1}{12n^2} \le 10^{-6}$
Solving for n,
$|E(T_n)| \le \frac{1}{12}10^6 \le n^2$
$n \ge \sqrt{\frac{1}{12}10^6}$
≈ 288.67

(b) $f(x) = \ln x$. Hence, $f'(x) = \frac{1}{x}, f''(x) = -\frac{1}{x^2}$. Therefore $|f''(x)| \le 1$.
The error using Midpoint Rule is

$|E(M_n)| \le 1\dfrac{(2-1)^3}{24n^2} \le 10^{-6}$

$|E(M_n)| \le \dfrac{1}{24n^2} \le 10^{-6}$

Solving for n,

$|E(M_n)| \le \dfrac{1}{24}10^6 \le n^2$

$n \ge \sqrt{\dfrac{1}{24}10^6}$

≈ 204.12

(c) $f(x) = \ln x$. Hence, $f'(x) = \dfrac{1}{x}$, $f''(x) = -\dfrac{1}{x^2}$, $f'''(x) = \dfrac{2}{x^3}$ and $f^{(4)}(x) = -\dfrac{6}{x^4}$. Therefore $\left|f^{(4)}(x)\right| \le 6$.

The error using Simpson's Rule is

$|E(S_n)| \le 6\dfrac{(2-1)^4}{180n^4} \le 10^{-6}$

$|E(S_n)| \le \dfrac{1}{30n^4} \le 10^{-6}$

Solving for n,

$|E(S_n)| \le \dfrac{1}{30}10^6 \le n^4$

$n \ge \sqrt[4]{\dfrac{1}{30}10^6}$

≈ 13.5

26. (a) $f(x) = x\ln x$. Hence, $f'(x) = 1+\ln x$ and $f''(x) = \dfrac{1}{x}$. Therefore $|f''(x)| \le 1$.

$|E(T_n)| \le 1\dfrac{(4-1)^3}{12n^2} \le 10^{-6}$

$|E(T_n)| \le \dfrac{27}{12n^2} \le 10^{-6}$

Solving for n,

$|E(T_n)| \le \dfrac{27}{12}10^6 \le n^2$

$n \ge \sqrt{\dfrac{27}{12}10^6}$

$= 1500.$

(b) $f(x) = x\ln x$. Hence, $f'(x) = 1+\ln x$, $f''(x) = \dfrac{1}{x}$. Therefore $|f''(x)| \le 1$.

The error using Trapezoidal Rule is

$|E(M_n)| \le 1\dfrac{(4-1)^3}{24n^2} \le 10^{-6}$

$|E(M_n)| \le \dfrac{27}{24n^2} \le 10^{-6}$

Solving for n,

$|E(M_n)| \le \dfrac{27}{24}10^6 \le n^2$

$n \ge \sqrt{\dfrac{27}{24}10^6}$

≈ 1060.66

(c) $f(x) = x\ln x$. Hence, $f'(x) = 1+\ln x$, $f''(x) = \dfrac{1}{x}$, $f'''(x) = -\dfrac{1}{x^2}$ and $f^{(4)}(x) = \dfrac{2}{x^3}$.

Therefore $\left|f^{(4)}(x)\right| \le 2$.

The error using Simpson's Rule is

$|E(S_n)| \le 2\dfrac{(4-1)^4}{180n^4} \le 10^{-6}$

$|E(S_n)| \le \dfrac{9}{10n^4} \le 10^{-6}$

Solving for n,

$|E(S_n)| \le \dfrac{9}{10}10^6 \le n^4$

$n \ge \sqrt[4]{\dfrac{9}{10}10^6}$

≈ 30.8

27. (a) $f(x) = e^{x^2}$. Hence, $f'(x) = 2xe^{x^2}$, $f''(x) = 2e^{x^2}\left(2x^2+1\right)$. Therefore, $|f''(x)| \le 6e \approx 16.3097$.

The error using Trapezoidal Rule is

$|E(T_n)| \le 16.3097\dfrac{(1-0)^3}{12n^2} \le 10^{-6}$

$|E(T_n)| \le \dfrac{16.3097}{12n^2} \le 10^{-6}$

Solving for n,

$|E(T_n)| \le \dfrac{16.3097}{12}10^6 \le n^2$

$n \ge \sqrt{\dfrac{16.3097}{12}10^6}$

$\approx 1165.$

(b) $f(x) = e^{x^2}$. Hence, $f'(x) = 2xe^{x^2}$, $f''(x) = 2e^{x^2}\left(2x^2+1\right)$. Therefore, $|f''(x)| \le 6e \approx 16.3097$.

The error using Trapezoidal Rule is

$|E(M_n)| \le 16.3097\dfrac{(1-0)^3}{24n^2} \le 10^{-6}$

$|E(M_n)| \le \dfrac{16.3097}{24n^2} \le 10^{-6}$

Solving for n,

$|E(M_n)| \le \dfrac{16.3097}{24}10^6 \le n^2$

$n \ge \sqrt{\dfrac{16.3097}{24}10^6}$

≈ 824.36

(c) $f(x) = e^{x^2}$. Hence,

$f'(x) = 2xe^{x^2}$,

$f''(x) = 2e^{x^2}\left(2x^2+1\right)$,

$f'''(x) = 4e^{x^2}\left(2x^3+3x\right)$

$f^{(4)}(x) = 4e^{x^2}\left(4x^4+12x^2+3\right)$.

Therefore, $|f''(x)| \le 76e \approx 206.5823$.
The error using Simpson's Rule is
$|E(S_n)| \le 206.5823 \frac{(1-0)^4}{180n^4} \le 10^{-6}$
$|E(S_n)| \le \frac{206.5823}{180n^2} \le 10^{-6}$
Solving for n,
$|E(S_n)| \le \frac{206.5823}{180} 10^6 \le n^2$
$n \ge \sqrt[4]{\frac{206.5823}{180} 10^6}$
≈ 32.7307.

28. (a) $f(x) = xe^x$
Hence,
$f'(x) = e^x(x+1)$
$f''(x) = e^x(x+2)$
Therefore,
$|f''(x)| \le 4e^2 \approx 21.21$
The error using Midpoint Rule is
$|E(M_n)| \le 21.21 \frac{(2-1)^3}{24n^2} \le 10^{-6}$
$|E(M_n)| \le \frac{21.21}{24n^2} \le 10^{-6}$
Solving for n,
$|E(M_n)| \le \frac{2402.0293}{24} 10^6 \le n^2$
$n \ge \sqrt{\frac{21.21}{24} 10^6}$
≈ 940.0797838

(b) $f(x) = xe^x$
Hence,
$f'(x) = e^x(x+1)$
$f''(x) = e^x(x+2)$
Therefore,
$|f''(x)| \le 4e^2 \approx 21.21$
The error using Trapezoidal Rule is
$|E(T_n)| \le 21.21 \frac{(2-1)^3}{12n^2} \le 10^{-6}$
$|E(T_n)| \le \frac{21.21}{12n^2} \le 10^{-6}$
Solving for n,
$|E(T_n)| \le \frac{21.21}{12} 10^6 \le n^2$
$n \ge \sqrt{\frac{21.21}{12} 10^6}$
≈ 1329.473580

(c) $f(x) = xe^x$
Hence,
$f'(x) = e^x(x+1), f''(x) = e^x(x+2)$
$f'''(x) = e^x(x+3)$
$f^{(4)}(x) = e^x(x+4)$
Therefore,
$\left|f^{(4)}(x)\right| \le 6e^2 \approx 31.82$
The error using Simpson's Rule is
$|E(S_n)| \le 31.82 \frac{(2-1)^4}{180n^4} \le 10^{-6}$
$|E(S_n)| \le \frac{31.82}{180n^2} \le 10^{-6}$
Solving for n,
$|E(S_n)| \le \frac{31.82}{180} 10^6 \le n^2$
$n \ge \sqrt[4]{\frac{31.82}{180} 10^6}$
≈ 20.50486515

29. We use $K = 60$, $L = 120$

n	EM_n	*Error Bound*
10	8.3×10^{-3}	2.5×10^{-2}

n	ET_n	*Error Bound*
10	1.6×10^{-2}	5×10^{-2}

n	ES_n	*Error Bound*
10	7.0×10^{-5}	6.6×10^{-3}

30. We use $K = L = 1$.

n	EM_n	*Error Bound*
10	0	1.3×10^{-2}

n	ET_n	*Error Bound*
10	0	2.6×10^{-2}

n	ES_n	*Error Bound*
10	0	1.7×10^{-4}

31. (a) Left Endpoints:
$\int_0^2 f(x)dx$
$\approx \frac{2-0}{4}[f(0) + f(.5) + f(1)$
$+f(1.5)]$
$= \frac{1}{2}(1 + .25 + 0 + .25)$
$= .75$

(b) Midpoint Rule:
$\int_0^2 f(x)dx$
$\approx \frac{2-0}{4}[f(.25) + f(.75)$
$+f(1.25) + f(1.75)]$
$= \frac{1}{2}(.65 + .15 + .15 + .65)$
$= .7$

(c) Trapezoidal Rule:
$\int_0^2 f(x)dx$

$\approx \dfrac{2-0}{2(4)}\,[f(0)+2f(.5)+2f(1)$
$+2f(1.5)+f(2)]$
$= \dfrac{1}{4}(1+.5+0+.5+1)$
$= .75$

(d) Simpson's rule: $\displaystyle\int_0^2 f(x)\,dx$
$= \dfrac{2}{12}\,[f(0)+4f(0.5)+2f(1)$
$+4f(1.5)+f(2)]$
$= \dfrac{1}{6}\,[1+4(0.25)+2(0)+4(0.25)+1]$
$= \dfrac{1}{6}\,[4]$
$= 0.66666$

32. (a) Left Endpoints:
$\displaystyle\int_0^2 f(x)dx$
$\approx \dfrac{1}{2}\,(f(0)+f(.5)+f(1)+f(1.5))$
$= \dfrac{1}{2}\,(0.5+0.8+0.5+0.1)$
$= 0.95$

(b) Midpoint Rule:
$\displaystyle\int_0^2 f(x)\,dx$
$\approx \dfrac{1}{2}\,(0.7+0.8+0.4+0.2)$
$= 1.05$

(c) Trapezoidal Rule:
$\displaystyle\int_0^2 f(x)dx$
$\approx \dfrac{1}{4}[0.5+2(0.8)+2(0.5)+2(0.1)$
$+\,0.5]$
$= 0.95$

(d) Simpson's rule: $\displaystyle\int_0^2 f(x)\,dx$
$= \dfrac{2-0}{12}\,[f(0)+4f(0.5)+2f(1)$
$+4f(1.5)+f(2)]$
$= \dfrac{1}{6}\,[0.5+4(0.9)+2(0.5)+4(0.1)+0.5]$
$= \dfrac{1}{6}\,[0.5+3.6+1+0.4+0.5]$
$= 1$

33. (a) Midpoint Rule:
$M_n < \int_a^b f(x)dx$

(b) Trapezoidal Rule:
$T_n > \int_a^b f(x)dx$

(c) Simpson's Rule:
Not enough information.

34. (a) Midpoint Rule:
$M_n < \int_a^b f(x)dx$

(b) Trapezoidal Rule:
$T_n > \int_a^b f(x)dx$

(c) Simpson's Rule:
$S_n \geq \int_a^b f(x)dx$

35. (a) Midpoint Rule:
$M_n > \int_a^b f(x)dx$

(b) Trapezoidal Rule:
$T_n < \int_a^b f(x)dx$

(c) Simpson's Rule:
Not enough information.

36. (a) Midpoint Rule: $M_n > \int_a^b f(x)dx$

(b) Trapezoidal Rule: $T_n < \int_a^b f(x)dx$

(c) Simpson's Rule: $S_n \leq \int_a^b f(x)dx$

37. (a) Midpoint Rule: $M_n < \int_a^b f(x)dx$

(b) Trapezoidal Rule: $T_n > \int_a^b f(x)dx$

(c) Simpson's Rule: $S_n = \int_a^b f(x)dx$

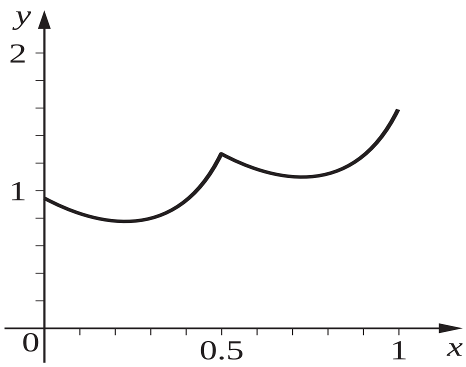
38. (a) Midpoint Rule: $M_n = \int_a^b f(x)dx$

(b) Trapezoidal Rule: $T_n = \int_a^b f(x)dx$

(c) Simpson's Rule: $S_n = \int_a^b f(x)dx$

39. $\frac{1}{2}(R_L + R_R)$
$= \sum\limits_{i=0}^{n-1} f(x_i) + \sum\limits_{i=1}^{n} f(x_i)$
$= f(x_0) + \sum\limits_{i=1}^{n-1} f(x_i) + \sum\limits_{i=1}^{n-1} f(x_i) + f(x_n)$
$= f(x_0) + 2\sum\limits_{i=1}^{n-1} f(x_i) + f(x_n) = T_n$

40.



41. $I_1 = \int\limits_0^1 \sqrt{1-x^2}dx$ is one fourth of the area of a circle with radius 1, so $\int\limits_0^1 \sqrt{1-x^2}dx = \frac{\pi}{4}$

$I_2 = \int\limits_0^1 \frac{1}{1+x^2}dx = \arctan x|_0^1$
$= \arctan 1 - \arctan 0 = \frac{\pi}{4}$

n	$S_n(\sqrt{1-x^2})$	$S_n(\frac{1}{1+x^2})$
4	0.65652	0.78539
8	0.66307	0.78539

The second integral $\int \frac{1}{1+x^2}dx$ provides a better algorithm for estimating π.

42. $\int\limits_{-h}^{h} \left(Ax^2 + Bx + c\right) dx$
$= \left(\frac{A}{3}x^3 + \frac{B}{2}x^2 + cx\right)\Big|_{-h}^{h}$
$= \frac{2}{3}Ah^3 + 2Ch$
$= \frac{h}{3}\left(2Ah^2 + 6C\right)$
$= \frac{h}{3}\left[f(-h) + 4f(0) + f(h)\right]$

43. (a) $\int\limits_{-1}^{1} xdx = 0$
$\left(-\frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}}\right) = 0$

(b) $\int\limits_{-1}^{1} x^2dx = \frac{2}{3}$
$\left(-\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^3 = \frac{2}{3}$

(c) $\int\limits_{-1}^{1} x^3dx = 0$
$\left(-\frac{1}{\sqrt{3}}\right)^3 + \left(\frac{1}{\sqrt{3}}\right)^3 = 0$

44. Simpson's Rule with $n = 2$:
$\int\limits_{-1}^{1} \pi\cos\left(\frac{\pi x}{2}\right)dx$
$\approx \frac{2}{6}\left(f(-1) + 4f\left(\frac{-1}{3}\right) + f(1)\right)$
$= \frac{1}{3}\left[\pi\cos\left(\frac{-\pi}{2}\right) + 4\pi\cos\left(\frac{-\pi}{6}\right) + \pi\cos\left(\frac{\pi}{2}\right)\right]$
$= \frac{\pi}{3}\left(0 + 2\sqrt{3} + 0\right) = \frac{2\pi}{\sqrt{3}}$
≈ 3.6276

Gaussian quadrature:
$\int\limits_{-1}^{1} \pi\cos\left(\frac{\pi x}{2}\right)dx$
$\approx f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$
$= \pi\cos\left(-\frac{\pi}{2\sqrt{3}}\right) + \pi\cos\left(\frac{\pi}{2\sqrt{3}}\right)$
≈ 3.87164

45. Simpson's Rule is not applicable because $\frac{\sin x}{x}$ is not defined at $x = 0$. $L = \lim\limits_{x\to 0}\frac{\sin x}{x}$
$= \lim\limits_{x\to 0}\frac{\cos x}{1} = \cos 0 = 1$
The two functions $f(x)$ and $\frac{\sin x}{x}$ differ only at one point,so $\int\limits_0^{\pi} f(x)\,dx = \int\limits_0^{\pi}\frac{\sin x}{x}dx$ We can now apply Simpson's Rule with $n = 2$:
$\int\limits_0^{\pi} f(x)\,dx$
$\approx \frac{\pi}{6}\left(1 + \frac{4\sin\pi}{\frac{\pi}{2}} + \frac{\sin\pi}{\pi}\right)$
$= \frac{\pi}{2}\left(\frac{1}{3} + \frac{8}{3\pi}\right)$

$\approx \frac{\pi}{2} \cdot 1.18$

46. The function $\frac{\sin x}{x}$ is not defined at $x = 0$, and it is symmetric across the y-axis. We define a new function

$f(x) = \begin{cases} \sin x/x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

over the interval $[0, \pi/2]$, and $\int_{-\pi/2}^{\pi/2} \frac{\sin x}{x} dx =$

$2 \int_0^{\pi/2} f(x)dx$

Use Simpson's Rule on $n = 2$:

$\int_0^{\pi/2} f(x)dx$

$\approx \frac{\pi}{12}\left(1 + \frac{\frac{\sqrt{2}}{2}}{\pi/4} + \frac{1}{\pi/2}\right)$

$\approx \frac{\pi}{2} \cdot 15.22$

Hence

$\int_{-\pi/2}^{\pi/2} \frac{\sin x}{x} dx \approx \frac{\pi}{2} \cdot 30.44$

47. Let I be the exact integral. Then we have

$T_n - I \approx -2(M_n - I)$
$T_n - I \approx 2I - 2M_n$
$T_n + 2M_n \approx 3I$
$\frac{T_n}{3} + \frac{2}{3}M_n \approx I$

48. The text does not say this, but we want to show that

$\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}$

In this case, we have data points: $x_0, x_1, x_2, x_3, ..., x_{2n}$.

The midpoint rule will use the points: $x_1, x_3, ..., x_{2n-1}$.

The trapezoidal rule will use the points: $x_0 x_2, ..., x_{2n}$.

$\frac{1}{3}T_n + \frac{2}{3}M_n$

$= \left(\frac{1}{3}\right)\left(\frac{b-a}{2n}\right)[f(x_0) + 2f(x_2) + 2f(x_4)$
$+ ... + 2f(x_{2n-2}) + f(x_{2n})]$
$+ \left(\frac{2}{3}\right)\left(\frac{b-a}{n}\right) \times [f(x_1) + f(x_3)$
$+ f(x_5) + ... + f(x_{2n-1}) + f(x_{2n})]$
$= \left(\frac{b-a}{2n}\right)[f(x_0) + 4f(x_1) + 2f(x_2)$
$+ 4f(x_3) + 2f(x_4) + ... + 2f(x_{2n-2})$
$+ 4f(x_{2n-1}) + f(x_{2n})]$
$= S_{2n}$

49. $f(x) + f(1-x)$

$= \frac{x^2}{2x^2 - 2x + 1} + \frac{(1-x)^2}{2(1-x)^2 - 2(1-x) + 1}$

$= \frac{x^2}{2x^2 - 2x + 1}$
$+ \frac{(1-x)^2}{2(1 - 2x + x^2) - 2 + 2x + 1}$

$= \frac{x^2}{2x^2 - 2x + 1} + \frac{(1-x)^2}{2x^2 - 2x + 1}$

$= \frac{x^2}{x^2 + (x-1)^2} + \frac{(1-x)^2}{(1-x)^2 + x^2}$

$= \frac{x^2 + (1-x)^2}{x^2 + (1-x)^2}$

$= 1$

By Trapezoidal Rule,

$\int_0^1 f(x)\,dx$

$= \frac{(1-0)}{2n}[f(x_0) + 2f(x_1)$
$+ 2f(x_2) + ... + 2f(x_{n-1}) + f(x_n)]$

$= \frac{(1-0)}{2n}\Big[f(0) + 2f(\frac{1}{n})$
$+ 2f(\frac{2}{n}) + ... + 2f(\frac{n-1}{n}) + f(1)\Big]$

as $f(x) + f(1-x) = 1$,

we have,

$f(0) + f(1) = 1,$
$f(\frac{1}{n}) + f(\frac{n-1}{n}) = 1$
$f(\frac{2}{n}) + f(\frac{n-2}{n}) = 1$
.
.
.
$f(\frac{n-1}{n}) + f(\frac{1}{n}) = 1$

Adding the above n equations, we get

$\left[f(0) + 2f(\frac{1}{n}) + .. + 2f(\frac{n-1}{n}) + f(1)\right] = n$

Hence,

$\int_0^1 f(x)\,dx = \frac{1}{2n}(n) = \frac{1}{2}$

50. $\int_0^n x^n dx$

$= \left(\frac{n-0}{2n}\right)[f_0 + 2f_1 + 2f_2 + ... + 2f_{n-1} + f_n]$

$= \frac{1}{2}[f(0) + 2f(1) + 2f(2) + 2f(3) + ...$
$+ 2f(n-1) + f(n)]$

$= \frac{1}{2}[(n^n) + 2(1^n + 2^n + 3^n + ... + (n-1)^n]$

Now

$$\int_0^n x^n dx = \left.\frac{x^{n+1}}{n+1}\right|_0^n$$

$$= \frac{n^{n+1}}{n+1}$$

The sum of the areas of the trapezoids is greater than the area defined by the curve over the interval 0 to n. $\dfrac{n^{n+1}}{n+1} < \dfrac{n^n}{2} + 1^n + 2^n + 3^n + ... + (n-1)^n$

$$\frac{n^{n+1}}{n+1} + \frac{n^n}{2} < 1^n + 2^n + +... + (n-1)^n + n^n$$

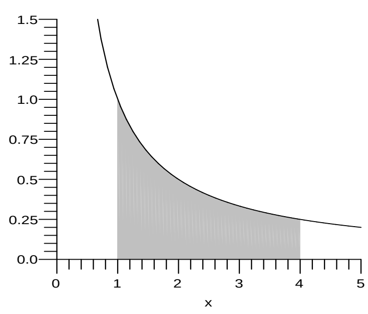
$$\frac{2n^{n+1} + n^{n+1} + n^n}{2(n+1)} < 1^n + 2^n + ... + n^n$$

$$\frac{3n^{n+1} + n^n}{2(n+1)} < 1^n + 2^n + 3^n + ... + n^n$$

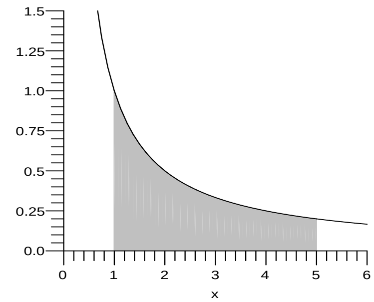
$$\frac{(3n+1)}{2(n+1)}n^n < 1^n + 2^n + 3^n + ... + n^n$$

4.8 The Natural Logarithm As An Integral

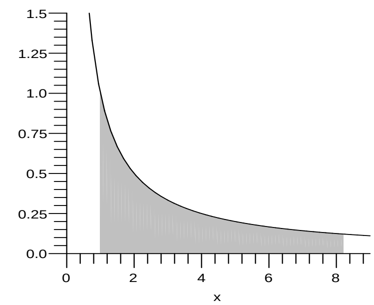
1. $\ln 4 = \ln 4 - \ln 1 = \ln x|_1^4 = \displaystyle\int_1^4 \frac{dx}{x}$



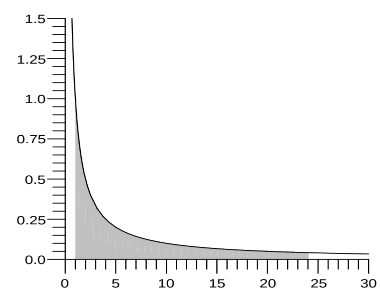
2. $\ln 5 = \displaystyle\int_1^5 \frac{dx}{x}$



3. $\ln 8.2 = \displaystyle\int_1^{8.2} \frac{dx}{x}$



4. $\ln 24 = \displaystyle\int_1^{24} \frac{dx}{x}$



5. $\ln 4 = \displaystyle\int_1^4 \frac{dx}{x}$
$\approx \dfrac{3}{12}\left(\dfrac{1}{1} + 4\dfrac{1}{1.75} + 2\dfrac{1}{1.5} + 4\dfrac{1}{3.25} + \dfrac{1}{4}\right)$
≈ 1.3868

6. $\ln 5 = \displaystyle\int_1^5 \frac{dx}{x}$
$\approx \dfrac{4}{12}\left(\dfrac{1}{1} + 4\dfrac{1}{2} + 2\dfrac{1}{3} + 4\dfrac{1}{4} + \dfrac{1}{5}\right)$
≈ 1.6108

7. (a) Simpson's Rule with $n = 32$:
$\ln 4 = \int_1^4 \dfrac{dx}{x} \approx 1.386296874$

(b) Simpson's Rule with $n = 64$:
$\ln 4 = \int_1^4 \frac{dx}{x} \approx 1.386294521$

8. (a) Simpson's Rule with $n = 32$:
$\ln 4 = \int_1^4 \frac{dx}{x} \approx 1.609445754$

(b) Simpson's Rule with $n = 64$:
$\ln 4 = \int_1^4 \frac{dx}{x} \approx 1.609438416$

9. $\frac{7}{2} \ln 2$

10. $\ln 2$

11. $\ln \left(\frac{3^2 \cdot \sqrt{3}}{9} \right) = \frac{1}{2} \ln 3$

12. $\ln \left(\frac{\frac{1}{9} \cdot \frac{1}{9}}{3} \right) = -5 \ln 3$

13. $\frac{1}{\sqrt{x^2+1}} \cdot \frac{1}{2} \left(x^2+1\right)^{-\frac{1}{2}} .2x$

14. $\frac{5x^4 \sin x \cos x + x^5 \cos^2 x - x^5 \sin x}{x^5 \sin x \cos x}$

15. $\frac{x^5+1}{x^4} \cdot \frac{4x^3\left(x^5+1\right) - x^4\left(5x^4\right)}{\left(x^5+1\right)^2}$

16. $\sqrt{\frac{x^5+1}{x^3}} \cdot \frac{1}{2} \cdot \left(\frac{x^3}{x^5+1} \right)^{-1/2}$
$\cdot \frac{3x^2\left(x^5+1\right) - x^3\left(5x^4\right)}{\left(x^5+1\right)^2}$

17. $\frac{d}{dx} \frac{1}{2} \left(\frac{\ln\left(x^2+1\right)}{\ln 7} \right)$
$= \frac{1}{2 \ln 7} \frac{d}{dx} \left(\ln\left(x^2+1\right) \right)$
$= \frac{1}{\ln 7} \left(\frac{x}{x^2+1} \right)$

18. $\frac{d}{dx} \left(\frac{x \ln 2}{\ln 10} \right) = \frac{\ln 2}{\ln 10} \frac{d}{dx}(x) = \log_{10} 2$

19. Let $y = 3^{\sin x}$
On taking natural logarithm.
$\ln y = \ln\left(3^{\sin x}\right) = \sin x \ln 3$
$\frac{1}{y} \frac{dy}{dx} = \frac{d}{dx}(\sin x \ln 3) = \ln 3 \frac{d}{dx}(\sin x)$
$\frac{1}{y} \frac{dy}{dx} = (\ln 3) \cos x$
$\frac{dy}{dx} = y (\ln 3) \cos x$
$\frac{dy}{dx} = 3^{\sin x} (\ln 3) \cos x$

20. $y = 4^{\sqrt{x}}$
On taking natural logarithm.
$\ln y = \ln\left(4^{\sqrt{x}}\right) = \sqrt{x} \ln 4$
$\frac{1}{y} \frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x} \ln 4\right)$
$= (\ln 4) \frac{d}{dx}\left(\sqrt{x}\right)$
$= (\ln 4) \left(\frac{1}{2\sqrt{x}} \right)$
$\frac{dy}{dx} = y \frac{(\ln 4)}{2\sqrt{x}}$
$\frac{dy}{dx} = \frac{4^{\sqrt{x}} (\ln 4)}{2\sqrt{x}}$

21. $\int \frac{1}{x \ln x} dx = \ln |\ln x| + c$

22. $\int \frac{1}{\sqrt{1-x^2} \sin^{-1} x} dx = \ln \left|\sin^{-1} x\right| + c$

23. Let $u = x^2, du = 2x dx$
$\int x 3^{x^2} dx = \frac{1}{2} \int 3^u du = \frac{3^{x^2}}{2 \ln 3} + c$

24. Let $u = 2^x,\ du = 2^x (\ln 2) dx$
$\int 2^x \sin\left(2^x\right) dx = \frac{1}{\ln 2} \int \sin(u)\, du$
$= \frac{-\cos\left(2^x\right)}{\ln 2} + c$

25. Let $u = \frac{2}{x},\ du = \left(\frac{-2}{x^2} \right) dx$
$\int \frac{e^{2/x}}{x^2} dx = -\frac{1}{2} \int e^u du$
$= -\frac{1}{2e^u} + c = -\frac{1}{2} e^{2/x} + c$

26. Let $u = \ln x^3, du = \left(\frac{3}{x} \right) dx$
$\int \frac{\sin\left(\ln x^3\right)}{x} dx = \frac{1}{3} \int \sin u du$
$= -\frac{1}{3} \cos u + c$
$= -\frac{1}{3} \cos\left(\ln x^3\right) + c$

27. $\int_0^1 \frac{x^2}{x^3-4} dx$
$= \frac{1}{3} \ln \left|x^3 - 4\right|\Big|_0^1$
$= \frac{1}{3} \ln 3 - \frac{1}{3} \ln 4 = \frac{1}{3} \ln \frac{3}{4}$

28. $\int_0^1 \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$
$= \ln \left|e^x + e^{-x}\right|\Big|_0^1$

$= \ln\left(e+e^{-1}\right) - \ln 2$
$= \ln\left(\frac{e+e^{-1}}{2}\right)$

29. $\int_0^1 \tan x dx = \int_0^1 \frac{\sin x}{\cos x} dx$
$= -\ln|\cos x||_0^1$
$= -\ln|\cos 1| - \ln|\cos 0|$
$= -\ln(\cos 1)$

30. Let $u = \ln x$, $du = \frac{dx}{x}$
$\int \frac{\ln x}{x} dx = \int u dx = \frac{u^2}{2} + c$
$= \frac{(\ln x)^2}{2} + c$
$\int_1^2 \frac{\ln x}{x} dx = \left.\frac{(\ln x)^2}{2}\right|_1^2$
$= \frac{\ln^2 2}{2} - \frac{\ln^2 1}{2} = \frac{\ln^2 2}{2}$

31. $\ln\left(\frac{a}{b}\right) = \ln\left(a \cdot \frac{1}{b}\right) = \ln a + \ln\left(\frac{1}{b}\right)$
$= \ln a - \ln b$

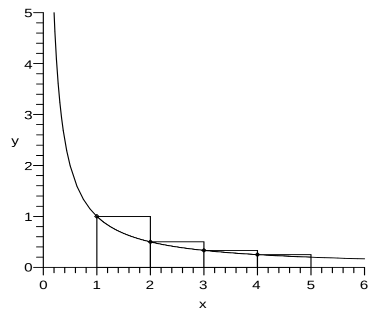
32. Consider $x = 2^{-n}$, where n is any integer for $x > 0$.

On taking natural logarithm.

$\ln x = \ln 2^{-n}$
$\Rightarrow \ln x = -n \ln 2$
Now $x \to 0, 2^{-n} \to 0 \Rightarrow n \to \infty$
$\Rightarrow \lim_{x\to 0_+} (\ln x) = \lim_{n\to\infty} (-n \ln 2)$
$= -(\ln 2) \lim_{n\to\infty} (n)$.
But, $\ln 2 \approx 0.6931$ and $\lim_{n\to\infty} n = \infty$
$\Rightarrow \lim_{x\to 0_+} (\ln x) = -\infty$.

33. We know that by definition, $\ln(n) = \int_1^n \frac{1}{x} dx$

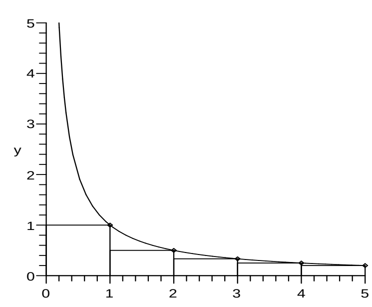
which is the area bounded by the curve $y = \frac{1}{x}$, the positive x-axis between the ordinates $x = 1$ and $x = n$. Let $y = f(x) = \frac{1}{x}$.



From the graph, it may be observed that the area bounded by $y = \frac{1}{x}$; the x-axis between the ordinates $x = 1$ and $x = n$ is lesser than the shaded area which is the sum of areas of the $(n-1)$ rectangles having width 1 unit and height $f(i)$
Thus from the graph,

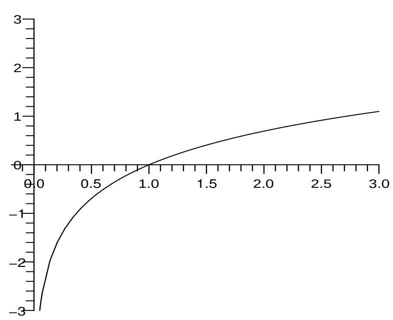
$\int_1^n \frac{1}{x} dx < \sum_{i=1}^{n-1} (f(i) \times 1)$
$\ln(n) < f(1) + f(2) + f(3) + \ldots$
$.. + f(n-1)$
or $\ln(n) < 1 + \frac{1}{2} + \ldots.. + \frac{1}{n-1}$
Hence proved. We know that,
$\lim_{n\to\infty} \ln(n) = \infty$
$\lim_{n\to\infty} \left(1 + \frac{1}{2} + \ldots.. + \frac{1}{n-1}\right) \geq \lim_{n\to\infty} \ln(n)$
$= \infty$

34. We know that by definition,
$\ln(n) = \int_1^n \frac{1}{x} dx$
which is the area bounded by the curve $y = \frac{1}{x}$, the positive x-axis between the ordinates $x = 1$ and $x = n$.
Let $y = f(x) = \frac{1}{x}$.



Let us consider $(n-1)$ rectangles, having width 1 unit and height $f(i+1)$ where $i = 1, 2, 3, ..., n-1$. Thus from the graph,
$\int_1^n \frac{1}{x} dx > \sum_{i=1}^{n-1} (f(i+1) \times 1)$
$\ln(n) > f(2) + f(3) + ... + f(n)$
or $\ln(n) > \frac{1}{2} + \frac{1}{3} ..... + \frac{1}{n}$.

35. Since the domain of the function $y = \ln x$ is $(0, \infty)$, $f'(x) = \frac{1}{x} > 0$ for $x > 0$. So f is increasing throughout the domain. Similarly, $f''(x) = -\frac{1}{x^2} < 0$ for $x > 0$. Therefore, the graph is concave down everywhere, the graph of the function $y = \ln x$ is as below.



36. Proof of (ii)

By using the rules of logarithm we have,
$\ln\left(\frac{e^r}{e^s}\right) = \ln(e^r) - \ln(e^s)$
$= r \ln e - s \ln e = r - s = \ln\left(e^{r-s}\right)$
Since $\ln x$ is one to one, it follows that
$\frac{e^r}{e^s} = e^{r-s}$.

Proof of (iii)

By using the rules of logarithm we have,
$\ln(e^r)^t = t \ln(e^r) = rt \ln e = \ln\left(e^{rt}\right)$
Since $\ln x$ is one to one, it follows that
$(e^r)^t = e^{rt}$.

37. $h = \ln e^h = \int_1^{e^h} \frac{1}{x} dx = \frac{e^h - 1}{\bar{x}}$,
for some $\bar{x}$ in $(0, h)$
$\frac{e^h - 1}{h} = \bar{x}$
as $h \to 0^+, \bar{x} \to 0$, then
$\lim_{h \to 0^+} \frac{e^h - 1}{h} = 0$
$-h = \ln e^{-h} = \int_1^{e^{-h}} \frac{1}{x} dx = \frac{e^{-h} - 1}{\bar{x}}$,
for some $\bar{x}$ in $(-h, 0)$
$\frac{e^{-h} - 1}{-h} = \bar{x}$
as $h \to 0^+, -h \to 0^-, \bar{x} \to 0$, then
$\lim_{h \to 0^+} \frac{e^{-h} - 1}{-h} = 0$

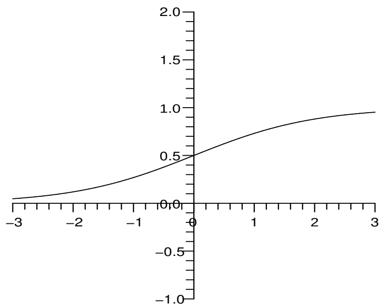
38. $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$ and $f'(1) = 1$.
On the other hand
$f'(a) = \lim_{x \to a} \frac{\ln x - \ln a}{x - a}$
$f'(1) = \lim_{x \to 1} \frac{\ln x - \ln 1}{x - 1} = 1$
$\lim_{x \to 1} \frac{\ln x}{x - 1} = 1$
Thus the reciprocal of $\frac{\ln x}{x - 1}$ has the same limit,
$\lim_{x \to 1} \frac{x - 1}{\ln x} = 1$
Substituting $x = e^h$, $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

39. (a) Given that, $y = \ln(x+1)$ by using a linear approximation.
$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$
For small value of x,
$f(x) \approx f(0) + f'(0)(x - 0)$
$\ln(1 + x) \approx 0 + 1 \cdot (x - 0)$
$\ln(1 + x) \approx x$.

(b) By using area under the curve.
Area the rectangle
$= f(1) \cdot x = x$
Also, $\int_1^{1+x} \frac{1}{t} dt = \ln t|_1^{1+x}$
$= \ln(1 + x) - \ln(1)$
$= \ln(1 + x)$.
As x approaches to zero, we get:
$\ln(1 + x) \approx x$

40. $f(x) = \ln x - 1$
$f'(x) = \frac{1}{x}$
$x_0 = 3$
$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{\ln 3 - 1}{\frac{1}{3}}$
$= 6 - 3 \ln 3 \approx 2.704163133$
$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 2.718245098$
$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 2.718281827$
$e \approx 2.718282183$
Three steps are needed to start at $x_0 = 3$ and obtain five digits of accuracy.

41. $f(x) = \dfrac{1}{1+e^{-x}}$



Using $\lim\limits_{x\to\infty} e^{-x} = 0$ we get
$\lim\limits_{x\to\infty} \dfrac{1}{1+e^{-x}} = 1.$
Using $\lim\limits_{x\to-\infty} e^{-x} = \infty$ we get
$\lim\limits_{x\to\infty} \dfrac{1}{1+e^{-x}} = 0.$
The function $f(x)$ is increasing over $(-\infty, \infty)$ and when $x = 0$,
$f(0) = \dfrac{1}{1+1} = \dfrac{1}{2}.$

So $g(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \ge 0 \end{cases}$

The threshold value for $g(x)$ to switch is $x = 0$. One way of modifying the function to move the threshold to $x = 4$ is to let $f(x) = \dfrac{1}{1+e^{-(x-4)}}$.

42. $1 - (9/10)^10 \approx 0.65132$
$1 - (19/20)^20 \approx 0.64151$
$1 - (9/10)^10 > 1 - (19/20)^20$

The probability of winning is lower.

When taking the limit as $n \to \infty$,

$$\lim_{n\to\infty} \left[1 - \left(\frac{n-1}{n}\right)^n\right]$$
$$= 1 - \lim_{n\to\infty} \left(\frac{n-1}{n}\right)^n$$
$$= 1 - \lim_{n\to\infty} \left(1 + \frac{-1}{n}\right)^n$$
$$= 1 - e^{-1}$$

43. $s(x) = x^2 \ln(1/x)$
$s'(x) = 2x \ln 1/x + x^2 \cdot x \cdot (-1/x^2)$
$= 2x \ln(1/x) - x = x(2\ln(1/x) - 1)$

$s'(x) = 0$ gives
$x = 0$ (which is impossible) or
$\ln(1/x) = 1/2, x = e^{-1/2}.$

Since $s'(x) \begin{cases} < 0 & \text{if } x < e^{-1/2} \\ > 0 & \text{if } x > e^{-1/2} \end{cases}$

The value $x = e^{-1/2}$ maximizes the transmission speed.

44. $\ln\left[\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n\right]$

$$= \lim_{n\to\infty} \ln\left(1 + \frac{1}{n}\right)^n$$
$$= \lim_{n\to\infty} n \ln\left(1 + \frac{1}{n}\right)$$
$$= \lim_{n\to\infty} \frac{\ln(1+1/n)}{1/n}$$
$$= \lim_{n\to\infty} \frac{-1/n^2}{-1/n^2(1+1/n)}$$
$$= \lim_{n\to\infty} \frac{1}{1+1/n}$$
$$= 1$$

Ch. 4 Review Exercises

1. $\displaystyle\int (4x^2 - 3)\,dx = \frac{4}{3}x^3 - 3x + c$

2. $\displaystyle\int (x - 3x^5)\,dx = \frac{x^2}{2} - \frac{1}{2}x^6 + c$

3. $\displaystyle\int \frac{4}{x}\,dx = 4\ln|x| + c$

4. $\displaystyle\int \frac{4}{x^2}\,dx = -\frac{4}{x} + c$

5. $\displaystyle\int 2\sin 4x\,dx = -\frac{1}{2}\cos 4x + c$

6. $\displaystyle\int 3\sec^2 x\,dx = 3\tan x + c$

7. $\displaystyle\int (x - e^{4x})\,dx = \frac{x^2}{2} - \frac{1}{4}e^{4x} + c$

8. $\displaystyle\int 3\sqrt{x}\,dx = 2x^{3/2} + c$

9. $\displaystyle\int \frac{x^2+4}{x}\,dx = \int (x + 4x^{-1})\,dx$
$\displaystyle= \frac{x^2}{2} + 4\ln|x| + c$

10. $\displaystyle\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\ln(x^2+4) + c$

11. $\displaystyle\int e^x(1 - e^{-x})\,dx = \int (e^x - 1)\,dx$
$= e^x - x + c$

12. $\int e^x(1+e^x)^2\,dx$
$= \int (e^x + 2e^{2x} + e^{3x})\,dx$
$= e^x + e^{2x} + \frac{1}{3}\,e^{3x} + c$

13. Let $u = x^2+4$, then $du = 2x\,dx$ and
$\int x\sqrt{x^2+4}\,dx$
$= \frac{1}{2}\int u^{1/2}\,du = \frac{1}{3}\,u^{3/2} + c$
$= \frac{1}{3}\,(x^2+4)^{3/2} + c$

14. $\int x(x^2+4)\,dx = \int (x^3+4x)\,dx$
$= \frac{x^4}{4} + 2x^2 + c$

15. Let $u = x^3$, $du = 3x^2\,dx$
$\int 6x^2\cos x^3\,dx = 2\int \cos u\,du$
$= 2\sin u + c = 2\sin x^3 + c$

16. Let $u = x^2$, $du = 2x\,dx$
$\int 4x\sec x^2\tan x^2\,dx$
$= 2\int \sec u\tan u\,du$
$= 2\sec u + c = 2\sec x^2 + c$

17. Let $u = 1/x$, $du = -1/x^2\,dx$
$\int \frac{e^{1/x}}{x^2}\,dx = -\int e^u\,du$
$= -e^u + c = -e^{1/x} + c$

18. Let $u = \ln x$, $du = \,dx/x$
$\int \frac{\ln x}{x}\,dx = \int u\,du$
$= \frac{u^2}{2} + c = \frac{(\ln x)^2}{2} + c$

19. $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$
$= -\ln|\cos x| + c$

20. Let $u = 3x+1, du = 3\,dx$
$\int \sqrt{3x+1}dx = \frac{1}{3}\int u^{1/2}\,du$
$= \frac{1}{3}\cdot\frac{2}{3}u^{3/2} + c = \frac{2}{9}(3x+1)^{3/2} + c$

21. $f(x) = \int (3x^2+1)\,dx = x^3 + x + c$
$f(0) = c = 2$
$f(x) = x^3 + x + 2$

22. $f(x) = \int e^{-2x}\,dx = -\frac{1}{2}e^{-2x} + c$
$f(0) = -\frac{1}{2} + c = 3$
$c = \frac{7}{2}$
$f(x) = -\frac{1}{2}e^{-2x} + \frac{7}{2}$

23. $s(t) = \int (-32t+10)\,dt$
$= -16t^2 + 10t + c$
$s(0) = c = 2$
$s(t) = -16t^2 + 10t + 2$

24. $v(t) = \int 6\,dt = 6t + c_1$
$v(0) = c_1 = 10$
$v(t) = 6t + 10$
$s(t) = \int (6t+10)\,dt = 3t^2 + 10t + c_2$
$s(0) = c_2 = 0$
$s(t) = 3t^2 + 10t$

25. $\sum_{i=1}^{6}(i^2+3i)$
$= (1^2+3\cdot 1) + (2^2+3\cdot 2) + (3^2+3\cdot 3)$
$+ (4^2+3\cdot 4) + (5^2+3\cdot 5) + (6^2+3\cdot 6)$
$= 4 + 10 + 18 + 28 + 40 + 54$
$= 154$

26. $\sum_{i=1}^{12} i^2 = 650$

27. $\sum_{i=1}^{100}(i^2-1)$
$= \sum_{i=1}^{100} i^2 - \sum_{i=1}^{100} 1$
$= \frac{100(101)(201)}{6} - 100$
$= 338,250$

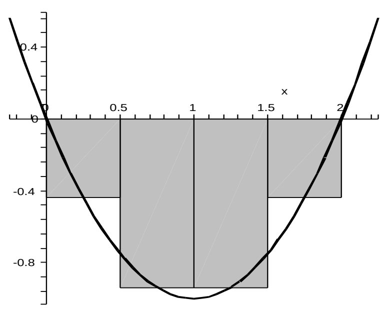
28. $\sum_{i=1}^{100}(i^2+2i)$
$= \sum_{i=1}^{100} i^2 + 2\cdot\sum_{i=1}^{100} i$
$= \frac{100(101)(201)}{6} + 100(101)$
$= 348,450$

29. $\frac{1}{n^3}\sum_{i=1}^{n}(i^2-i)$
$= \frac{1}{n^3}\left(\sum_{i=1}^{n} i^2 - \cdot\sum_{i=1}^{n} i\right)$

$$= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right)$$
$$= \frac{(n+1)(2n+1)}{6n^2} - \frac{n+1}{2n^2}$$
$$\lim_{n\to\infty} \frac{1}{n^3}\sum_{i=1}^{n}(i^2 - i)$$
$$= \lim_{n\to\infty}\left(\frac{(n+1)(2n+1)}{6n^2} - \frac{n+1}{2n^2}\right)$$
$$= \frac{2}{6} - 0 = \frac{1}{3}$$

30. Evaluation points: $0.25, 0.75, 1.25, 1.75$

Riemann sum $= \Delta x \sum_{i=1}^{n} f(c_i)$
$$= \frac{2}{4}\sum_{i=1}^{4}(c_i^2 - 2c_i)$$
$$= \frac{1}{2}\big[(0.25^2 - 2\cdot 0.25) + (0.75^2 - 2\cdot 0.75) + (1.25^2 - 2\cdot 1.25) + (1.75^2 - 2\cdot 1.75)\big]$$
$$= -2.75$$



31. Riemann sum $= \frac{2}{8}\sum_{i=1}^{8} c_i^2 = 2.65625$

32. Riemann sum $= \frac{2}{8}\sum_{i=1}^{8} c_i^2 = 0.6875$

33. Riemann sum $= \frac{3}{8}\sum_{i=1}^{8} c_i^2 \approx 4.668$

34. Riemann sum $= \frac{1}{8}\sum_{i=1}^{8} c_i^2 \approx 0.6724$

35.

(a) Left-endpoints:
$$\int_0^{1.6} f(x)\,dx$$
$$\approx \frac{1.6-0}{8}(f(0) + f(.2) + f(.4) + f(.6) + f(.8) + f(1) + f(1.2) + f(1.4))$$
$$= \frac{1}{5}(1 + 1.4 + 1.6 + 2 + 2.2 + 2.4 + 2 + 1.6)$$
$$= 2.84$$

(b) Right-endpoints:
$$\int_0^{1.6} f(x)\,dx$$
$$\approx \frac{1.6-0}{8}(f(.2) + f(.4) + f(.6) + f(.8) + f(1) + f(1.2) + f(1.4) + f(1.6))$$
$$= \frac{1}{5}(1.4 + 1.6 + 2 + 2.2 + 2.4 + 2 + 1.6 + 1.4)$$
$$= 2.92$$

(c) Trapezoidal Rule:
$$\int_0^{1.6} f(x)\,dx$$
$$\approx \frac{1.6-0}{2(8)}[f(0) + 2f(.2) + 2f(.4) + 2f(.6) + 2f(.8) + 2f(1) + 2f(1.2) + 2f(1.4) + f(1.6)]$$
$$= 2.88$$

(d) Simpson's Rule:
$$\int_0^{1.6} f(x)\,dx$$
$$\approx \frac{1.6-0}{3(8)}[f(0) + 4f(.2) + 2f(.4) + 4f(.6) + 2f(.8) + 4f(1) + 2f(1.2) + 4f(1.4) + f(1.6)]$$
$$\approx 2.907$$

36.

(a) Left-endpoints:
$$\int_1^{4.2} f(x)\,dx$$
$$\approx (0.4)[f(1.0) + f(1.4) + f(1.8) + f(2.2) + f(2.6) + f(3.0) + f(3.4) + f(3.8)]$$
$$= (0.4)(4.0 + 3.4 + 3.6 + 3.0 + 2.6 + 2.4 + 3.0 + 3.6)$$
$$= 10.24$$

(b) Right-endpoints:
$$\int_1^{4.2} f(x)\,dx$$
$$\approx (0.4)[f(1.4) + f(1.8) + f(2.2) + f(2.6) + f(3.0) + f(3.4) + f(3.8) + f(4.2)]$$
$$= (0.4)(3.4 + 3.6 + 3.0 + 2.6$$

$+ 2.4 + 3.0 + 3.6 + 3.4)$
$= 10.00$

(c) Trapezoidal Rule:
$$\int_1^{4.2} f(x)\,dx$$
$$\approx \frac{0.4}{2}[f(1.0) + 2f(1.4) + 2f(1.8) + 2f(2.2) + 2f(2.6) + 2f(3.0) + 2f(3.4) + 2f(3.8) + f(4.2)]$$
$$= (0.2)[4.0 + 2(3.4) + 2(3.6) + 2(3.0) + 2(2.6) + 2(2.4) + 2(3.0) + 2(3.6) + 3.4]$$
$$= 10.12$$

(d) Simpson's Rule:
$$\int_1^{4.2} f(x)\,dx$$
$$\approx \frac{0.4}{3}[f(1.0) + 4f(1.4) + 2f(1.8) + 4f(2.2) + 2f(2.6) + 4f(3.0) + 2f(3.4) \quad + 4f(3.8) + f(4.2)]$$
$$= \frac{0.4}{3}[4.0 + 4(3.4) + 2(3.6) + 4(3.0) + 2(2.6) + 4(2.4) + 2(3.0) + 4(3.6) + 3.4]$$
$$\approx 10.05333$$

37. See Example 7.10.

Simpson's Rule is expected to be most accurate.

38. In this situation, the Midpoint Rule will be less than the actual integral. The Trapezoid Rule will be an overestimate.

39. We will compute the area A_n of n rectangles using right endpoints. In this case $\Delta x = \frac{1}{n}$ and $x_i = \frac{i}{n}$
$$A_n = \sum_{i=1}^{n} f(x_i)\Delta x = \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)$$
$$= \frac{1}{n}\sum_{i=1}^{n} 2\cdot\left(\frac{i}{n}\right)^2$$
$$= \frac{2}{n^3}\sum_{i=1}^{n} i^2$$
$$= \left(\frac{2}{n^3}\right)\frac{n(n+1)(2n+1)}{6}$$
$$= \frac{(n+1)(2n+1)}{3n^2}$$
Now, to find the integral, we take the limit:
$$\int_0^1 x^2\,dx = \lim_{n\to\infty} A_n$$
$$= \lim_{n\to\infty}\frac{(n+1)(2n+1)}{3n^2}$$
$$= \frac{2}{3}$$

40. We will compute the area A_n of n rectangles using right endpoints. In this case $\Delta x = \dfrac{2}{n}$ and $x_i = \dfrac{2i}{n}$
$$A_n = \sum_{i=1}^{n} f(x_i)\Delta x = \frac{2}{n}\sum_{i=1}^{n} f\left(\frac{2i}{n}\right)$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)^2 + 1$$
$$= \frac{8}{n^3}\sum_{i=1}^{n} i^2 + \frac{2}{n}\sum_{i=1}^{n} 1$$
$$= \left(\frac{8}{n^3}\right)\frac{n(n+1)(2n+1)}{6} + \left(\frac{2}{n}\right)n$$
$$= \frac{4(n+1)(2n+1)}{3n^2} + 2$$
Now, to find the integral, we take the limit:
$$\int_0^2 (x^2+1)\,dx = \lim_{n\to\infty} A_n$$
$$= \lim_{n\to\infty}\left(\frac{4(n+1)(2n+1)}{3n^2} + 2\right)$$
$$= \frac{8}{3} + 2 = \frac{14}{3}$$

41. Area $= \displaystyle\int_0^3 (3x - x^2)\,dx$
$$= \left(\frac{3x^2}{2} - \frac{x^3}{3}\right)\Big|_0^3 = \frac{9}{2}$$

42. Area
$$= \int_0^1 (x^3 - 3x^2 + 2x)\,dx - \int_1^2 (x^3 - 3x^2 + 2x)\,dx$$
$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

43. The velocity is always positive, so distance traveled is equal to change in position.
$$\text{Dist} = \int_1^2 (40 - 10t)\,dt$$
$$= (40t - 5t^2)\Big|_1^2 = 25$$

44. The velocity is always positive, so distance traveled is equal to change in position.
$$\text{Dist} = \int_0^2 20e^{-t/2}\,dt = (-40e^{-t/2})\Big|_0^2$$
$$= 40(-e^{-1} + 40) \approx 25.2848$$

45. $f_{ave} = \frac{1}{2}\int_0^2 e^x\, dx = \frac{e^2-1}{2} \approx 3.19$

46. $f_{ave} = \frac{1}{4}\int_0^4 (4x - x^2)\, dx = \frac{8}{3}$

47. $\int_0^2 (x^2-2)\, dx = \left(\frac{x^3}{3} - 2x\right)\Big|_0^2 = -\frac{4}{3}$

48. $\int_{-1}^1 (x^3 - 2x)\, dx = \left(\frac{x^4}{4} - x^2\right)\Big|_{-1}^1 = 0$

49. $\int_0^{\pi/2} \sin 2x\, dx = -\frac{1}{2}\cos 2x\Big|_0^{\pi/2} = 1$

50. $\int_0^{\pi/4} \sec^2 x\, dx = \tan x\Big|_0^{\pi/4} = 1$

51. $\int_0^{10} (1 - e^{-t/4})\, dt$
$= \left(t + 4e^{-t/4}\right)\Big|_0^{10} = 6 + 4e^{-5/2}$

52. $\int_0^1 te^{-t^2}\, dt$
$= \left(-\frac{1}{2}e^{-t^2}\right)\Big|_0^1 = -\frac{1}{2}\left(e^{-1} - 1\right)$

53. $\int_0^2 \frac{x}{x^2+1}\, dx = \frac{1}{2}\ln|x^2+1|\Big|_0^2$
$= \frac{\ln 5}{2}$

54. $\int_1^2 \frac{\ln x}{x}\, dx = \left(\frac{\ln^2 x}{2}\right)\Big|_1^2 = \frac{\ln^2 2}{2}$

55. $\int_0^2 x\sqrt{x^2+4}\, dx$
$= \left(\frac{1}{2}\cdot\frac{2}{3}\cdot(x^2+4)^{3/2}\right)\Big|_0^2$
$= \frac{16\sqrt{2}-8}{3}$

56. $\int_0^2 x(x^2+1)\, dx$
$= \left(\frac{1}{4}(x^2+1)^2\right)\Big|_0^2 = 6$

57. $\int_0^1 (e^x - 2)^2\, dx = \int_0^1 (e^{2x - 4e^x + 4})\, dx$
$= \left(\frac{1}{2}e^{2x} - 4e^x + 4x\right)\Big|_0^2$
$= \left(\frac{e^2}{2} - 4e + 4\right) - \left(\frac{1}{2} - 4\right)$
$= \frac{e^2}{2} - 4e + \frac{15}{2}$

58. $\int_{-\pi}^{\pi} \cos(x/2)\, dx$
$= (2\sin(x/2))\Big|_{-\pi}^{\pi} = 4$

59. $f'(x) = \sin x^2 - 2$

60. $f'(x) = \sqrt{(x^2)^2+1}\cdot 2x$

61.

a) Midpoint Rule:
$\int_0^1 \sqrt{x^2+4}\, dx$
$\approx \frac{1-0}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right)\right.$
$\left. +f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right]$
≈ 2.079

b) Trapezoidal Rule:
$\int_0^1 \sqrt{x^2+4}\, dx$
$\approx \frac{1-0}{2(4)}\left[f(0) + 2f\left(\frac{1}{4}\right)\right.$
$+2f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)$
$\left. +f(1)\right]$
≈ 2.083

c) Simpson's Rule:
$\int_0^1 \sqrt{x^2+4}\, dx$
$\approx \frac{1-0}{3(4)}\left[f(0) + 4f\left(\frac{1}{4}\right)\right.$
$\left. +2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1)\right]$
≈ 2.080

62.

a) Midpoint Rule:
$\int_0^2 e^{-x^2/4}\, dx$
$\approx \frac{2}{4}[f(0.25) + f(0.75)$
$+ f(1.25) + f(1.75)]$
≈ 1.497494

b) Trapezoidal Rule:
$\int_0^2 e^{-x^2/4}\, dx$
$\approx \frac{2}{8}[f(0) + 2f(.5) + 2f(1)$

$+ 2f(1.5) + f(2)]$
≈ 1.485968

c) Simpson's Rule:

$$\int_0^2 e^{-x^2/4}\,dx \approx \frac{2}{12}[f(0) + 4f(.5) + 2f(1) + 4f(1.5) + f(2)] \approx 1.493711$$

63.

n	Midpoint	Trapezoid	Simpson's
20	2.08041	2.08055	2.08046
40	2.08045	2.08048	2.08046

64.

n	Midpoint	Trapezoid	Simpson's
20	1.493802	1.493342	1.493648
40	1.493687	1.493572	1.493648

65. Consider $u = \tanh\left(\frac{t}{2}\right) = \dfrac{\sinh\left(\frac{t}{2}\right)}{\cosh\left(\frac{t}{2}\right)}$

$$= \frac{\left(\frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{2}\right)}{\left(\frac{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}{2}\right)} = \frac{\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)}{\left(e^{\frac{t}{2}} + e^{-\frac{t}{2}}\right)}$$

therefore $\dfrac{1-u^2}{1+u^2} = \dfrac{1 - \left(\frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}\right)^2}{1 + \left(\frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}\right)^2}$

$$= \frac{\left(e^{\frac{t}{2}} + e^{-\frac{t}{2}}\right)^2 - \left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^2}{\left(e^{\frac{t}{2}} + e^{-\frac{t}{2}}\right)^2 + \left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^2}$$

$$= \frac{2\left(e^t + e^{-t}\right)}{4} = \cosh t\,,$$

similarly, $\dfrac{2u}{1+u^2} = \dfrac{2\frac{\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)}{\left(e^{\frac{t}{2}} + e^{-\frac{t}{2}}\right)}}{1 + \left(\frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}\right)^2}$

$$= \frac{2\left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)\left(e^{\frac{t}{2}} + e^{-\frac{t}{2}}\right)}{\left(e^{\frac{t}{2}} + e^{-\frac{t}{2}}\right)^2 + \left(e^{\frac{t}{2}} - e^{-\frac{t}{2}}\right)^2}$$

$$= \frac{2\left(e^t - e^{-t}\right)}{4} = \sinh t$$

(a) $\displaystyle\int \frac{1}{\sinh t + \cosh t}dt$

$$= \int \frac{1}{\frac{2u}{(1-u^2)} + \frac{(1+u^2)}{(1-u^2)}}du$$

(Put:$u = \tanh(t/2)$)

$$= \int \frac{\left(1-u^2\right)}{(1+u)^2}du$$

$$= \int \left(\frac{1-u}{1+u}\right)du$$

$$= \int \left(\frac{2}{1+u} - 1\right)du$$

$$= 2\ln(1+u) - u$$

$$= 2\ln(1 + \tanh(t/2)) - \tanh(t/2)$$

(b) $\displaystyle\int \frac{\sinh t + \cosh t}{1 + \cosh t}dt$

$$= \int \frac{\frac{2u}{(1-u^2)} + \frac{\left(1+u^2\right)}{(1-u^2)}}{1 + \frac{(1+u^2)}{(1-u^2)}}du$$

$$= \int \frac{(1+u)^2}{2}du$$

$$= \frac{1}{2}\left(\frac{(1+u)^3}{3}\right)$$

$$= \frac{(1 + \tanh(t/2))^3}{6}$$