الملف نموذج هيكل امتحان الفصل الثاني مع الحلول

# هيكل امتحانات نهاية الفصل 2 EoT2 Exam Coverage

| Maximum Overall Grade<br>العلامة القصوى الممكنة | Marks per Question<br>درجة كل سؤال | Type of Questions<br>طبيعة الاسئلة | Number of Question<br>عدد الاسئلة | Grade<br>الصف | Subject<br>المادة |
|---|---|---|---|---|---|
| 100 | 5 | MCQs<br>اختيار من متعدد | 25 | 12 Advanced<br>الثاني عشر المتقدم | Math<br>رياضيات |

| Mode of Implementation<br>طريقة التطبيق | Exam Duration<br>مدة الامتحان |
|---|---|
| swiftAssess | 120 minutes |

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| | |
|---|---|
| **Best 20 answers out of 25 will count.<br>Example: 14 correct answers yield a grade of 70/100, while 20 and 23 correct answers yield a (full) grade of 100/100 each.** | **تحتسب أفضل 20 إجابة من 25.<br>مثال 14 إجابة صحيحة تعطي علامة 70/100 بينما 20 أو 23 إجابة صحيحة تعطي العلامة الكاملة أي100/100** |
| **Questions might appear in a different order in the actual exam.** | **قد تظهر الأسئلة بترتيب مختلف في الامتحان الفعلي** |
| **As it appears in the textbook/LMS/SoW.** | **كما وردت في كتاب الطالب و الخطة الفصلية** |

أرجو مشاركتها مع قروبات الطلبة و فالكم التفوق و النجاح

تراعى الحلول الأخرى

| **Definition** |
|---|
| The linear (or tangent line) approximation of f(x) at $x = x_0$ is the function $L(x) = f(x_0) + f'(x_0)(x - x_0)$ |
| $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, for\ n + 0,1,2,3, \ldots$$ |

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| 1 | Find the linear approximation of a given function at a given point | 1 to 6 | 236 |

| 1-Find the linear approximation to f(x) at $x = x_0$ Use the linear approximation to estimate the given number | |
|---|---|
| $f(x) = \sqrt{x}, x_0 = 1, \sqrt{1.2}$ | |
| (a) $f(x) = \sqrt{x}, x_0 = 1$<br>$f(x_0) = f(1) = \sqrt{1} = 1$<br>$f'(x) = \frac{1}{2}x^{-1/2}$<br>$f'(x_0) = f'(1) = \frac{1}{2}$<br>So,<br>$L(x) = f(x_0) + f'(x_0)(x - x_0)$<br>$= 1 + \frac{1}{2}(x - 1)$<br>$= \frac{1}{2} + \frac{1}{2}x$ | (b) Using the approximation $L(x)$ to estimate $\sqrt{1.2}$, we get $\sqrt{1.2} = f(1.2) \approx L(1.2) = \frac{1}{2} + \frac{1}{2}(1.2) = 1.1$ |

| 2-Find the linear approximation to f(x) at $x = x_0$ Use the linear approximation to estimate the given number | |
|---|---|
| $f(x) = (x+1)^{\frac{1}{3}}, x_0 = 0, \sqrt[3]{1.2}$ | |
| (a) $f(x_0) = f(0) = 1$ and<br>$f'(x) = \frac{1}{3}(x+1)^{-2/3}$<br>So, $f'(0) = \frac{1}{3}$<br>The Linear approximation is,<br>$L(x) = 1 + \frac{1}{3}(x - 0) = 1 + \frac{1}{3}x$ | (b) Using the approximation $L(x)$ to estimate $\sqrt[3]{1.2}$, we get $\sqrt[3]{1.2} = f(0.2) \approx L(0.2) = 1 + \frac{1}{3}(0.2) = 1.066$ |

| 3-Find the linear approximation to f(x) at $x = x_0$ Use the linear approximation to estimate the given number | |
|---|---|
| $f(x) = \sqrt{2x+9}, x_0 = 0, \sqrt{8.8}$ | |
| (a) $f(x) = \sqrt{2x+9}, x_0 = 0$<br>$f(x_0) = f(0) = \sqrt{2 \cdot 0 + 9} = 3$<br>$f'(x) = \frac{1}{2}(2x+9)^{-1/2} \cdot 2$<br>$= (2x+9)^{-1/2}$<br>$f'(x_0) = f'(0) = (2 \cdot 0 + 9)^{-1/2} = \frac{1}{3}$<br>So,<br>$L(x) = f(x_0) + f'(x_0)(x - x_0)$<br>$= 3 + \frac{1}{3}(x - 0)$<br>$= 3 + \frac{1}{3}x$ | (b) Using the approximation $L(x)$ to estimate $\sqrt{8.8}$, we get $\sqrt{8.8} = f(-0.1) \approx L(-0.1) = 3 + \frac{1}{3}(-0.1) = 3 - 0.033 = 2.967$ |

| 4-Find the linear approximation to f(x) at $x = x_0$ Use the linear approximation to estimate the given number | |
|---|---|
| $f(x) = \frac{2}{x}, x_0 = 1, \frac{2}{0.99}$ | |
| (a) $f(x) = \frac{2}{x}$, $x_0 = 1$<br>$f(x_0) = f(1) = 2$<br>$f'(x) = -\frac{2}{x^2}$ and so $f'(1) = -2$<br>The linear approximation is<br>$L(x) = 2 + (-2)(x-1)$ | (b) Using the approximation $L(x)$ to estimate $\frac{2}{0.99}$, we get $\frac{2}{0.99} = f(0.99) \approx L(0.99) = 2 + (-2)(0.99 - 1) = 2.02$ |

| 5-Find the linear approximation to f(x) at $x = x_0$ Use the linear approximation to estimate the given number | |
|---|---|
| $f(x) = sin3x, x_0 = 0, \sin(0.3)$ | |
| (a) $f(x) = \sin 3x$, $x_0 = 0$<br>$f(x_0) = \sin(3 \cdot 0) = 0$<br>$f'(x) = 3\cos 3x$<br>$f'(x_0) = f'(0) = 3\cos(3 \cdot 0) = 3$<br>So,<br>$L(x) = f(x_0) + f'(x_0)(x - x_0)$<br>$= 0 + 3(x - 0)$<br>$= 3x$ | (b) Using the approximation $L(x)$ to estimate $\sin(0.3)$, we get $\sin(0.3) = f(0.1) \approx L(0.1) = 3(0.1) = 0.3$ |

| 6-Find the linear approximation to f(x) at $x = x_0$ Use the linear approximation to estimate the given number | |
|---|---|
| $f(x) = sinx, x_0 = \pi, \sin(3.0)$ | |
| (a) $f(x) = \sin x$, $x_0 = \pi$<br>$f(x_0) = \sin\pi = 0$<br>$f'(x) = \cos x$<br>$f'(x_0) = f'(\pi) = \cos\pi = -1$<br>The linear approximation is,<br>$L(x) = f(x_0) + f'(x_0)(x - x_0)$<br>$= 0 + (-1)(x - \pi) = \pi - x$ | (b) Using the approximation $L(x)$ to estimate $\sin(3.0)$, we get $\sin(3.0) = f(3.0) \approx L(3.0) = \pi - 3.0$ |

THEOREM 2.1 (L'Hopital's Rule)
Suppose that f and g are differentiable on the interval (a,b) , except possibly at the point $c \in (a,b)$ and $g'(x) \neq 0$ on (a,b), except possibly at c. Suppose further that $\lim_{x\to c}\frac{f(x)}{g(x)}$ has indeterminate form $\frac{0}{0}$ $or$ $\frac{\infty}{\infty}$ and that $\lim_{x\to c}\frac{f'(x)}{g'(x)} = L$ $or$ $\pm\infty$
Then $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$
##$\infty - \infty$ ,$0\infty, 0^0, 1^\infty, \infty^0$ change to form $\frac{0}{0}$ $or$ $\frac{\infty}{\infty}$

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1-Find the indicated limits

$$\lim_{x\to -2}\frac{x+2}{x^2-4}$$

$$\lim_{x\to -2}\frac{x+2}{x^2-4}$$
$$= \lim_{x\to -2}\frac{x+2}{(x+2)(x-2)}$$
$$= \lim_{x\to -2}\frac{1}{x-2} = -\frac{1}{4}$$

2-Find the indicated limits

$$\lim_{x\to 2}\frac{x^2-4}{x^2-3x+2}$$

$$\lim_{x\to 2}\frac{x^2-4}{x^2-3x+2}$$
$$= \lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)(x-1)}$$
$$= \lim_{x\to 2}\frac{x+2}{x-1} = 4$$

3-Find the indicated limits

$$\lim_{x\to \infty}\frac{3x^2+2}{x^2-4}$$

$$\lim_{x\to \infty}\frac{3x^2+2}{x^2-4}$$
$$= \lim_{x\to \infty}\frac{3+\frac{2}{x^2}}{1-\frac{4}{x^2}}$$
$$= \frac{3}{1} = 3$$

### 4-Find the indicated limits

$$\lim_{x \to -\infty} \frac{x+1}{x^2+4x+3}$$

$\lim\limits_{x \to -\infty} \dfrac{x+1}{x^2+4x+3}$ is type $\dfrac{\infty}{\infty}$;

we apply L'Hôpital's Rule to get

$\lim\limits_{x \to -\infty} \dfrac{1}{2x+4} = 0.$

### 5-Find the indicated limits

$$\lim_{t \to 0} \frac{e^{2t}-1}{t}$$

$\lim\limits_{t \to 0} \dfrac{e^{2t}-1}{t}$ is type $\dfrac{0}{0}$;

we apply L'Hôpital's Rule to get

$\lim\limits_{t \to 0} \dfrac{\frac{d}{dt}\left(e^{2t}-1\right)}{\frac{d}{dt}t}$

$\lim\limits_{t \to 0} \dfrac{2e^{2t}}{1} = \dfrac{2}{1} = 2$

### 6-Find the indicated limits

$$\lim_{t \to 0} \frac{\sin t}{e^{3t}-1}$$

$\lim\limits_{t \to 0} \dfrac{\sin t}{e^{3t}-1}$ is type $\dfrac{0}{0}$;

we apply L'Hôpital's Rule to get

$\lim\limits_{t \to 0} \dfrac{\frac{d}{dt}(\sin t)}{\frac{d}{dt}\left(e^{3t}-1\right)} = \lim\limits_{t \to 0} \dfrac{\cos t}{3e^{3t}} = \dfrac{1}{3}$

### 21-Find the indicated limits

$$\lim_{x \to \infty} \frac{\ln x}{x^2}$$

$\lim\limits_{x \to \infty} \dfrac{\ln x}{x^2}$ is type $\dfrac{\infty}{\infty}$

we apply L'Hôpital's Rule to get

$\lim\limits_{x \to \infty} \dfrac{1/x}{2x} = \lim\limits_{x \to \infty} \dfrac{1}{2x^2} = 0.$

### 22-Find the indicated limits

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$$

$\lim\limits_{x \to \infty} \dfrac{\ln x}{\sqrt{x}}$ is type $\dfrac{\infty}{\infty}$;

we apply L'Hôpital's Rule to get

$\lim\limits_{x \to \infty} \dfrac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim\limits_{x \to \infty} \dfrac{2}{\sqrt{x}} = 0.$

## 25-Find the indicated limits

$$\lim_{t\to 1}\frac{\ln(\ln t\ )}{\ln t}$$

$\lim\limits_{t\to 1}\frac{\ln(\ln t)}{\ln t}$
As $t$ approaches ln from below, $\ln t$ is a small negative number. Hence $\ln(\ln t)$ is undefined, so the limit is undefined.

## 29-Find the indicated limits

$$\lim_{x\to 0^+}\frac{\ln x}{\cot x}$$

$\lim\limits_{x\to 0^+}\frac{\ln x}{\cot x}$ is type $\frac{\infty}{\infty}$
we apply L’Hôpital’s Rule to get
$\lim\limits_{x\to 0^+}\frac{1/x}{-\csc^2 x}$
$=\lim\limits_{x\to 0^+}\left(-\sin x\cdot\frac{\sin x}{x}\right)=(0)(1)=0.$

## 30-Find the indicated limits

$$\lim_{x\to 0^+}\frac{\sqrt{x}}{\ln x}$$

$\lim\limits_{x\to 0^+}\frac{\sqrt{x}}{\ln x}=0$ (numerator goes to 0 and denominator goes to $-\infty$).

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5- Find all critical numbers by hand. Use your knowledge of the type of graph to determine whether the critical number represents a local maximum, local minimum or neither.

(a) $f(x) = x^3 - 3x^2 + 6x$

(a) $f(x) = x^3 - 3x^2 + 6x$
$f'(x) = 3x^2 - 6x + 6$
$3x^2 - 6x + 6 = 3(x^2 - 2x + 2) = 0$
We can use the quadratic formula to find the roots, which are $x = 1 \pm \sqrt{-1}$. These are imaginary so there are no real critical points.

5- Find all critical numbers by hand. Use your knowledge of the type of graph to determine whether the critical number represents a local maximum, local minimum or neither.

(b) $f(x) = -x^3 + 3x^2 - 3x$

(b) $f(x) = -x^3 + 3x^2 - 3x$

$$\begin{aligned} f'(x) &= -3x^2 + 6x - 3 \\ &= 3(-x^2 + 2x - 1) \\ &= -3(x^2 - 2x + 1) \\ &= -3(x-1)^2 \end{aligned}$$

$f'(x) = 3(x-1)^2 = 0$ when $x = 1$.
Since $f(x)$ is a cubic with only one critical number it is neither local min nor max.

6- Find all critical numbers by hand. Use your knowledge of the type of graph to determine whether the critical number represents a local maximum, local minimum or neither.

(a) $f(x) = x^4 - 2x^2 + 1$

(a) $f(x) = x^4 - 2x^2 + 1$

$$\begin{aligned} f'(x) &= 4x^3 - 4x \\ &= 4x(x^2 - 1) \\ &= 4x(x-1)(x+1) \end{aligned}$$

$f'(x) = 0$ when $x = 0, \pm 1$.
$x = 0, \pm 1$ are critical numbers. $x = 0$ is local maximum and $x = \pm 1$ are local minimum.

6- Find all critical numbers by hand. Use your knowledge of the type of graph to determine whether the critical number represents a local maximum, local minimum or neither.

(b) $f(x) = x^4 - 3x^2 + 2$

(b) $f(x) = x^4 - 3x^3 + 2$

$$\begin{aligned} f'(x) &= 4x^3 - 9x^2 \\ &= x^2(4x - 9) \end{aligned}$$

$f'(x) = 0$ when $x = 0, \frac{9}{4}$.
$x = 0, \frac{9}{4}$ are critical points. $x = \frac{9}{4}$ is local minimum and $x = 0$ is neither max nor min.

| 25- Find the absolute extrema of the given function on each indicated interval. |
|---|
| (a) $f(x) = x^3 - 3x + 1 \ on \ [0,2]$ |

$f(x) = x^3 - 3x + 1$
$f'(x) = 3x^2 - 3 = 3(x^2 - 1)$
$f'(x) = 0$ for $x = \pm 1$.

(a) On $[0, 2]$, 1 is the only critical number.
We calculate:
$f(0) = 1$
$f(1) = -1$ is the abs min.
$f(2) = 3$ is the abs max.

| 25- Find the absolute extrema of the given function on each indicated interval. |
|---|
| (b) $f(x) = x^3 - 3x + 1 \ on \ [-3,2]$ |

$f(x) = x^3 - 3x + 1$
$f'(x) = 3x^2 - 3 = 3(x^2 - 1)$
$f'(x) = 0$ for $x = \pm 1$.

(b) On the interval $[-3, 2]$, we have both 1 and $-1$ as critical numbers.
We calculate:
$f(-3) = -17$ is the abs min.
$f(-1) = 3$ is the abs max.
$f(1) = -1$
$f(2) = 3$ is also the abs max.

| 26- Find the absolute extrema of the given function on each indicated interval. |
|---|
| (a) $f(x) = x^4 - 8x^2 + 2 \ on \ [-3,1]$ |

$f(x) = x^4 - 8x^2 + 2$
$f'(x) = 4x^3 - 16x = 0$ when $x = 0$ and $x = \pm 2$.

(a) On $[-3, 1]$:
$f(-3) = 11$, $f(-2) = -14$, $f(0) = 2$, and $f(1) = -5$.
The abs min on this interval is $f(-2) = -14$ and the abs max is $f(-3) = 11$.

| 26- Find the absolute extrema of the given function on each indicated interval. |
|---|
| (b) $f(x) = x^4 - 8x^2 + 2 \ on \ [-1,3]$ |

$f(x) = x^4 - 8x^2 + 2$
$f'(x) = 4x^3 - 16x = 0$ when $x = 0$ and $x = \pm 2$.

(b) On $[-1, 3]$:
$f(-1) = -5$, $f(2) = -14$, and $f(3) = 11$.
The abs min on this interval is $f(2) = -14$ and the abs max is $f(3) = 11$.

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| 7 | Find the local extrema of a given function using the First Derivative Test | 13-14-25 | 267 |

13- Find all critical numbers by hand. Use the first Derivative Test to classify the location of a local maximum, local minimum or neither.

$$y = xe^{-2x}$$

$y = xe^{-2x}$
$y' = 1 \cdot e^{-2x} + x \cdot e^{-2x}(-2)$
$\quad = e^{-2x} - 2xe^{-2x}$
$\quad = e^{-2x}(1-2x)$
$x = \frac{1}{2}$
$e^{-2x}(1-2x) > 0$ on $(-\infty, 1/2)$
$e^{-2x}(1-2x) < 0$ on $(1/2, \infty)$
So $y = xe^{-2x}$ has a local maximum at $x = 1/2$.

14- Find all critical numbers by hand. Use the first Derivative Test to classify the location of a local maximum, local minimum or neither.

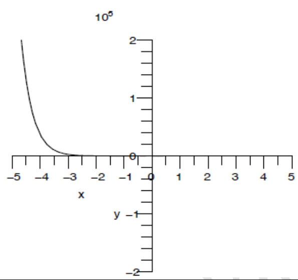
$$y = x^2 e^{-x}$$

$y = x^2e^{-x}$
$y' = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2-x)$.
At $x = 0$ the slope changes from negative to positive indicating a local minimum. At $x = 2$ the slope changes from positive to negative indicating a local maximum.

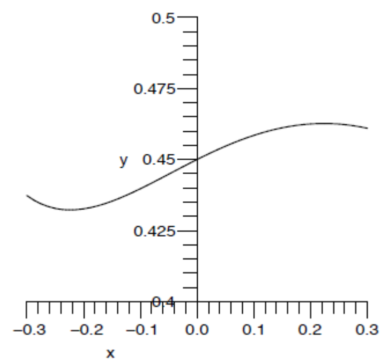
25- Approximate the x-coordinate of all extrema and sketch graphs showing global and local behavior of the function.

$$y = (x^2 + x + 0.45)\, e^{-2x}$$

$y' = (2x+1)e^{-2x} + (x^2+x+0.45)(-2)e^{-2x}$
Local min at $x = -0.2236$; local max at $x = 0.2236$.
Local behavior near $x = 0$ looks like




Global behavior of the function looks like



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| 6 | Identify increasing and decreasing functions | 45-46 | 276 |
| 8 | Learn the notion of an Inflection Point and find one | 1 to 5 | 276 |
| 9 | Determine the concavity of a function using the first and second derivatives | Example - 1 | 271 |

Example -1- Determine where the graph of $f(x) = 2x^3 + 9x^2 - 24x - 10$ is concave up and concave down, and draw a graph showing all significant features of the function

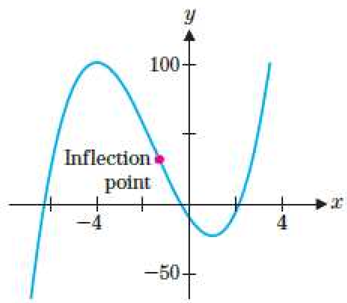
**Solution** Here, we have $f'(x) = 6x^2 + 18x - 24$

and from our work in example 4.3, we have

$$f'(x)\begin{cases} > 0 \text{ on } (-\infty, -4) \cup (1, \infty) & f \text{ increasing.} \\ < 0 \text{ on } (-4, 1). & f \text{ decreasing.} \end{cases}$$

Further, we have $f''(x) = 12x + 18 \begin{cases} > 0, \text{ for } x > -\frac{3}{2} & \text{Concave up.} \\ < 0, \text{ for } x < -\frac{3}{2}. & \text{Concave down.} \end{cases}$

Using all of this information, we are able to draw the graph shown in Figure 4.56. Notice that at the point $\left(-\frac{3}{2}, f\left(-\frac{3}{2}\right)\right)$, the graph changes from concave down to concave up. Such points are called *inflection points*, which we define more precisely in Definition 5.2. ■




1- Determine where the intervals where the graph of the given function is concave up and concave down, and identify inflection points $f(x) = x^3 - 3x^2 + 4x - 1$

$f'(x) = 3x^2 - 6x + 4$
$f''(x) = 6x - 6 = 6(x - 1)$
$f''(x) > 0$ on $(1, \infty)$
$f''(x) < 0$ on $(-\infty, 1)$
So $f$ is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.
$x = 1$ is a point of inflection.

2- Determine where the intervals where the graph of the given function is concave up and concave down, and identify inflection points $f(x) = x^4 - 6x^2 + 2x + 3$

$f'(x) = 4x^3 - 12x + 2$ and $f''(x) = 12x^2 - 12$. The graph is concave up where $f''(x)$ is positive, and concave down where $f''(x)$ is negative. Concave up for $x < -1$ and $x > 1$, and concave down for $-1 < x < 1$.
$x = -1, 1$ are points of inflection.

3- Determine where the intervals where the graph of the given function is concave up and concave down, and identify inflection points $f(x) = x + \frac{1}{x}$

$f(x) = x + \frac{1}{x} = x + x^{-1}$
$f'(x) = 1 - x^{-2}$
$f''(x) = 2x^{-3}$
$f''(x) > 0$ on $(0, \infty)$
$f''(x) < 0$ on $(-\infty, 0)$
So $f$ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$.
$x = 0$ is a point of inflection.

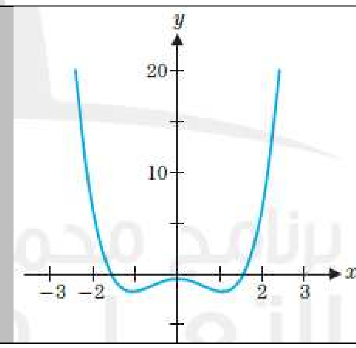
4- Determine where the intervals where the graph of the given function is concave up and concave down, and identify inflection points $f(x) = x + 3(1 - x)^{\frac{1}{3}}$

$y' = 1 - (1 - x)^{-2/3}$ and $y'' = \frac{-2}{3}(1 - x)^{-5/3}$.
Concave up for $x > 1$ and concave down for $x < 1$.
$x = 1$ is a point of inflection.

5- Determine where the intervals where the graph of the given function is concave up and concave down, and identify inflection points $f(x) = sinx - cosx$

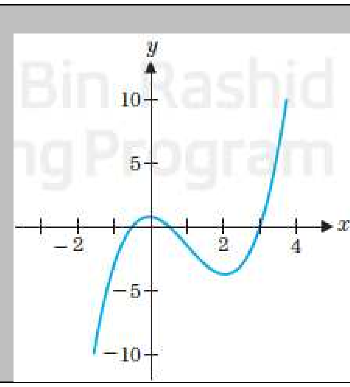
$f'(x) = \cos x + \sin x$
$f''(x) = -\sin x + \cos x$
$f''(x) < 0$ on $\ldots\left(\frac{\pi}{4}, \frac{5\pi}{4}\right) \cup \left(\frac{9\pi}{4}, \frac{13\pi}{4}\right)\ldots$
$f''(x) > 0$ on $\ldots\left(\frac{3\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, \frac{9\pi}{4}\right)\ldots$
$f$ is concave down on $\ldots\left(\frac{\pi}{4}, \frac{5\pi}{4}\right) \cup \left(\frac{9\pi}{4}, \frac{13\pi}{4}\right)\ldots$,
concave up on $\ldots\left(\frac{3\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, \frac{9\pi}{4}\right)\ldots$
$x = k\pi + \frac{\pi}{4}$ are the points of inflection for any interger $k$.

45- Estimate the intervals of increase and decrease the locations of local extrema, intervals of concavity and locations of inflection points




$f(x)$ is concave up on $(-\infty, -0.5)$ and $(0.5, \infty)$; $f(x)$ is concave down on $(-0.5, 0.5)$. $f(x)$ is decreasing on the intervals $(-\infty, 1)$ and $(0, 1)$ ; increasing on the intervals $(-1, 0)$ and $(1, \infty)$. $f(x)$ has local maxima at 0 and minima at -1 and 1. Inflection points of $f(x)$ are $-0.5$ and $0.5$.

46- Estimate the intervals of increase and decrease the locations of local extrema, intervals of concavity and locations of inflection points




$f(x)$ is concave up on $(1, \infty)$; $f(x)$ is concave down on $(-\infty, 1)$. $f(x)$ is increasing on the intervals $(-\infty, 0)$ and $(2, \infty)$; decreasing on the intervals $(0, 2)$. Inflection point of $f(x)$ is 1.

| Q | Learning Outcome*** | Example/Exercise | Page |
|---|---|---|---|
| 10 | Sketch the graph of a given function using its properties and its first and second derivative | 6 to10 | 286 |

**6- Graph the function and completely discuss the graph $f(x) = \frac{x^2-1}{x}$**

$f(x) = \frac{x^2-1}{x} = x - \frac{1}{x}$

There are $x$-intercepts at $x = \pm 1$, but no $y$-intercepts. The domain is $\{x|x \neq 0\}$.

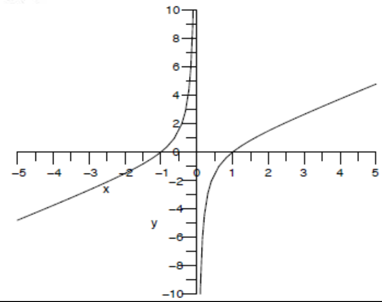
$f(x)$ has a vertical asymptote at $x = 0$ such that $f(x) \to \infty$ as $x \to 0^-$ and $f(x) \to -\infty$ as $x \to 0^+$.

$f'(x) = 1 + x^{-2} > 0$, So there is no critical numbers. $f(x)$ is increasing function.

$f''(x) = -2x^{-3}$

$f''(x) > 0$ on $(-\infty, 0)$ so $f(x)$ is concave up on this interval and $f''(x) < 0$ on $(0, \infty)$ so $f(x)$ is concave down on this interval, but $f(x)$ has an vertical asymptote (not an inflection point) at $x = 0$.

Finally, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.




**7- Graph the function and completely discuss the graph $f(x) = \frac{x^2+4}{x^3}$**

$f(x) = \frac{x^2+4}{x^3}$ has no $x$-intercept and no $y$-intercept. The domain of $f$ includes all real numbers $x \neq 0$. $f(x)$ has a vertical asymptote at $x = 0$

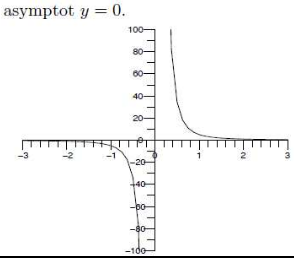
$$f'(x) = \frac{2x(x^3) - (x^2+4)(3x^2)}{(x^3)^2} = \frac{-(x^2+12)}{x^4}$$

Since $f'(x) = 0$ has no real roots, the graph has no extrema. $f'(x) < 0$ on $(-\infty, 0)$ and $(0, \infty)$ so $f(x)$ is decreasing on these intervals.

$$f''(x) = -\left[\frac{x^4(2x) - (x^2+12)(4x^3)}{(x^4)^2}\right] = \frac{2[x^2+24]}{x^5}$$

$f''(x) < 0$ on $(-\infty, 0)$ so $f(x)$ is concave down on this interval and $f''(x) > 0$ on $(0, \infty)$ so $f(x)$ is concave up on this interval, but $f(x)$ has an asymptote (not an inflection point) at $x = 0$.

Finally, $f(x) \to 0$ as $x \to -\infty$ and $f(x) \to 0$ as $x \to \infty$. Therefore, the graph has horizontal asymptot $y = 0$.



**8- Graph the function and completely discuss the graph $f(x) = \frac{x-4}{x^3}$**

$f(x) = \frac{x-4}{x^3}$

The graph has $x$-intercepts at $x = 4$, but no $y$-intercepts. The domain of $f$ includes all real numbers $x \neq 0$. $f(x)$ has a vertical asymptote at $x = 0$

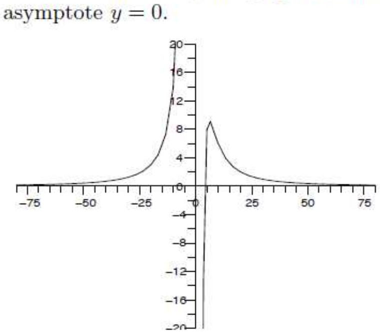
$$f'(x) = \frac{x^3 - (x-4)(3x^2)}{(x^3)^2} = \frac{-2x+12}{x^4}$$

The critical numbers is $x = 6$. We find that $f'(x) > 0$ on $(-\infty, 0)$ and $(0,6)$ so $f(x)$ is increasing on these intervals. $f'(x) < 0$ on $(6, \infty)$, so $f(x)$ is decreasing on these intervals. Therefore, the graph has a local maximum at $x = 6$.

$$f''(x) = \frac{(x^4)(-2) - (-2x+12)(4x^3)}{(x^4)^2} = \frac{6x-48}{x^5}$$

$f''(x) > 0$ on $(-\infty, 0)$ and $(8, \infty)$ so $f(x)$ is concave up on this interval and $f''(x) < 0$ on $(0, 8)$ so $f(x)$ is concave down on this interval, but $f(x)$ has an inflection point at $x = 8$.

Finally, $f(x) \to 0$ as $x \to -\infty$ and $f(x) \to 0$ as $x \to \infty$. Therefore, the graph has horizontal asymptote $y = 0$.



9- Graph the function and completely discuss the graph $f(x) = \frac{2x}{x^2-1}$

$f(x) = \frac{2x}{x^2-1}$

The graph has $x$-intercept and $y$-intercept at $(0, 0)$. The domain of $f$ includes all real numbers $x = \pm 1$. $f(x)$ has vertical asymptotes at $x = \pm 1$.

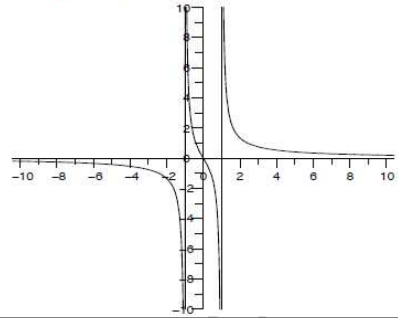
$$f'(x) = \frac{2(x^2-1) - (2x)(2x)}{(x^2-1)^2}$$
$$= \frac{-2(x^2+1)}{(x^2-1)^2}$$

Since $f'(x) = 0$ has no real roots, the graph has no extrema. $f'(x) < 0$ on $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$ and $(1, \infty)$ so $f(x)$ is decreasing on these intervals.

$$f''(x) = -2\left[\frac{2x(x^2-1)[x^2-1-2x^2-2]}{(x^2-1)^4}\right]$$
$$= \frac{4x[x^2+3]}{(x^2-1)^3}$$

$f''(x) > 0$ on $(-1, 0)$ and $(1, \infty)$ so $f(x)$ is concave up on this interval and $f''(x) < 0$ on $(-\infty, -1)$ and $(0, 1)$ so $f(x)$ is concave down on this interval, but $f(x)$ has an inflection point at $x = 0$.

Finally, $f(x) \to 0$ as $x \to -\infty$ and $f(x) \to 0$ as $x \to \infty$. Therefore, the graph has horizontal asymptote $y = 0$.




10- Graph the function and completely discuss the graph $f(x) = \frac{3x^2}{x^2+1}$

$f(x) = \frac{3x^2}{x^2+1}$

The graph has $x$-intercept and $y$-intercept at $(0, 0)$. The domain of $f$ includes all real numbers.

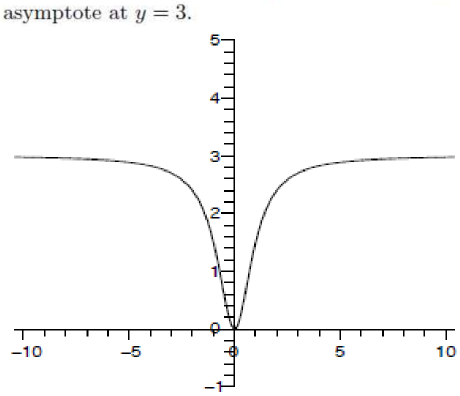
$$f'(x) = \frac{(x^2+1)(6x) - (3x^2)(2x)}{(x^2+1)^2}$$
$$= \frac{6x}{(x^2+1)^2}$$

$f'(x) < 0$ on $(-\infty, 0)$ so $f(x)$ is decreasing on these intervals and $f'(x) > 0$ on $(0, \infty)$ so $f(x)$ is increasing on these interval.

$$f''(x) = \frac{(x^2+1)[6(x^2+1) - 24x^2]}{(x^2+1)^4}$$
$$= \frac{6-18x^2}{(x^2+1)^3}$$

The critical numbers are $x = \pm\sqrt{\frac{1}{3}}$. We find that $f''(x) > 0$ on $\left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right)$ so $f(x)$ is concave up on this interval and we find that $f''(x) < 0$ on $\left(-\infty, -\sqrt{\frac{1}{3}}\right)$ and $\left(\sqrt{\frac{1}{3}}, \infty\right)$ so $f(x)$ is concave down on this interval, but the graph has inflection points at $x = \pm\sqrt{\frac{1}{3}}$.

Finally, $f(x) \to 3$ as $x \to -\infty$ and $f(x) \to 3$ as $x \to \infty$. Therefore, the graph has horizontal asymptote at $y = 3$.



| Q | Learning Outcome*** | Example/Exercise | Page |
|---|---|---|---|
| 11 | Solve mathematical and real-life optimization problems | 1 to7 | 296 |

1- A three-sided fence is to be built next to a straight section of river, which forms the fourth side of a rectangular region. The enclosed area is to equal 1800 $ft^2$. Find the minimum perimeter and the dimensions of the corresponding enclosure.

$A = xy = 1800$

$y = \frac{1800}{x}$

$P = 2x + y = 2x + \frac{1800}{x}$

$P' = 2 - \frac{1800}{x^2} = 0$

$2x^2 = 1800$

$x = 30$

$P'(x) > 0$ for $x > 30$

$P'(x) < 0$ for $0 < x < 30$

So $x = 30$ is min.

$y = \frac{1800}{x} = \frac{1800}{30} = 60$

So the dimensions are $30' \times 60'$ and the minimum perimeter is 120 ft.

2- A three-sided fence is to be built next to a straight section of river, which forms the fourth side of a rectangular region. There is 96 feet of fencing available. Find the maximum enclosed area and the dimensions of the corresponding enclosure.

If $y$ is the length of fence opposite the river, and $x$ is the length of the other two sides, then we have the constraint $2x + y = 96$. We wish to maximize

$A = xy = x(96 - 2x)$.

$A' = 96 - 4x = 0$ when $x = 24$.

$A'' = -4 < 0$ so this gives a maximum. Reasonable possible values of $x$ range from 0 to 48, and the area is 0 at these extremes. The maximum area is $A = 1152$, and the dimensions are $x = 24$, $y = 48$.

3- A two-pen corral is to be built. The outline of the corral forms two identical adjoining rectangles. The there is 120 feet of fencing available. What dimensions of the corral maximize the enclosed area?

$P = 2x + 3y = 120$

$3y = 120 - 2x$

$y = 40 - \frac{2}{3}x$

$A = xy$

$A(x) = x\left(40 - \frac{2}{3}x\right)$

$A'(x) = 1\left(40 - \frac{2}{3}x\right) + x\left(-\frac{2}{3}\right)$

$= 40 - \frac{4}{3}x = 0$

$40 = \frac{4}{3}x$

$x = 30$

$A'(x) > 0$ for $0 < x < 30$

$A'(x) < 0$ for $x > 30$.

So $x = 30$ is max, $y = 40 - \frac{2}{3} \cdot 30 = 20$.

So the dimensions are $20' \times 30'$.

| 4- A showroom for a department store is to be rectangular with walls on three sides, 6-ft door openings on the two facing sides and 10-ft door opening on remaining wall. The showroom is to have 800 $ft^2$ of floor space. What dimensions will minimize the length of wall used? |
|---|
| Let $x$ be the length of the sides facing each other and $y$ be the length of the third side. We have the constraint that $xy = 800$, or $y = 800/x$. We also know that $x > 6$ and $y > 10$. The function we wish to minimize is the length of walls needed, or the side length minus the width of the doors. $L = (y-10) + 2(x-6) = 800/x + 2x - 22$. $L' = -800/x^2 + 2 = 0$ when $x = 20$. $L'' = 1600/x^3 > 0$ when $x = 20$ so this is a minimum. Possible values of $x$ range from 6 to 80. $L(6) = 123.3$, $L(80) = 148$, and $L(20) = 58$. To minimize the length of wall, the facing sides should be 20 feet, and the third side should be 40 feet. |

5- Show that the rectangle of maximum area for a given perimeter P is always a square.

$$A = xy$$
$$P = 2x + 2y$$
$$2y = P - 2x$$
$$y = \frac{P}{2} - x$$
$$A(x) = x\left(\frac{P}{2} - x\right)$$
$$A'(x) = 1 \cdot \left(\frac{P}{2} - x\right) + x(-1)$$
$$= \frac{P}{2} - 2x = 0$$
$$P = 4x$$
$$x = \frac{P}{4}$$
$A'(x) > 0$ for $0 < x < P/4$
$A'(x) < 0$ for $x > P/4$

So $x = P/4$ is max,
$$y = \frac{P}{2} - x = \frac{P}{2} - \frac{P}{4} = \frac{P}{4}$$
So the dimensions are $\frac{P}{4} \times \frac{P}{4}$. Thus we have a square.

6- Show that the rectangle of minimum perimeter for a given area A is always a square.

We have a rectangle with sides $x$ and $y$ and area $A = xy$, and that we wish to minimize the perimeter,
$P = 2x + 2y = 2x + 2 \cdot \frac{A}{x}$.
$P' = 2 - \frac{2A}{x^2} = 0$ when $x = \sqrt{A}$.

$P'' = 4A/x^3 > 0$ here, so this is a minimum. Possible values of $x$ range from 0 to $\infty$. As $x$ approaches these values the perimeter grows without bound. For fixed area, the rectangle with minimum perimeter has dimensions $x = y = \sqrt{A}$, a square.

7- A box with no top is be built by taking a 6 in-by- 10 in sheet of cardboard, cutting x-in squares out of each corner and folding up the sides. Find the value of x that maximizes the volume of the box.

$$V = l \cdot w \cdot h$$
$$V(x) = (10-2x)(6-2x)\cdot x,\ 0 \le x \le 3$$
$$\begin{aligned} V'(x) &= -2(6-2x)\cdot x + (10-2x)(-2)\cdot x \\ &\quad + (10-2x)(6-2x) \\ &= 60 - 64x + 12x^2 \\ &= 4(3x^2 - 16x + 15) \\ &= 0 \end{aligned}$$

$$\begin{aligned} x &= \frac{16 \pm \sqrt{(-16)^2 - 4\cdot 3\cdot 15}}{6} \\ &= \frac{8}{3} \pm \frac{\sqrt{19}}{3} \\ x &= \frac{8}{3} + \frac{\sqrt{19}}{3} > 3. \end{aligned}$$

$V'(x) > 0$ for $x < 8/3 - \sqrt{19}/3$

$V'(x) < 0$ for $x > 8/3 - \sqrt{19}/3$

So $x = \frac{8}{3} - \frac{\sqrt{19}}{3}$ is a max.

| Q | Learning Outcome*** | Example/Exercise | Page |
|---|---|---|---|
| 12 | Solve mathematical and real-life problems on related rates | 1 to 7 | 303 |

1- Oil spills out of a tanker at the rate of 120 gl/min per minute. The oil spreads in a circle with a thickness of $\frac{1}{4}''$. Given that 1 $ft^3$ equals 7.5 gallons, determine the rate at which the radius of the spill is increasing when the radius reaches (a) 100 ft and (b) 200 ft. Explain why the rate decreases as the radius increases.

$V(t) = (\text{depth})(\text{area}) = \frac{\pi}{48}[r(t)]^2$
(units in cubic feet per min)

$V'(t) = \frac{\pi}{48}2r(t)r'(t) = \frac{\pi}{24}r(t)r'(t)$

We are given $V'(t) = \frac{120}{7.5} = 16.$

Hence $16 = \frac{\pi}{24}r(t)r'(t)$ so

$r'(t) = \frac{(16)(24)}{\pi r(t)}.$

(a) When $r = 100$,

$$r'(t) = \frac{(16)(24)}{100\pi} = \frac{96}{25\pi} \approx 1.2223 \text{ ft/min},$$

(b) When $r = 200$,

$$r'(t) = \frac{(16)(24)}{200\pi} = \frac{48}{25\pi} \approx 0.61115 \text{ ft/min}$$

2- Oil spills out of a tanker at the rate of 90 gallon per minute. The oil spreads in a circle with a thickness of $\frac{1}{8}''$. Determine the rate at which the radius of the spill is increasing when the radius reaches 100 ft

$V = (\text{depth})(\text{area})$. $\frac{1}{8}'' = \frac{1}{96}'$, so
$V(t) = \frac{1}{96}\pi r(t)^2$.

Differentiating we find $\frac{dV}{dt} = \frac{2\pi}{96}r(t)\frac{dr}{dt}$.

Using 1 $ft^3$ = 7.5 gal, the rate of change of volume is $\frac{90}{7.5} = 12$. So when $r(t) = 100$,

$12 = \frac{2\pi}{96}100\frac{dr}{dt}$, and

$\frac{dr}{dt} = \frac{144}{25\pi}$ feet per minute.

3- Oil spills out of a tanker at the rate of g gallon per minute. The oil spreads in a circle with a thickness of $\frac{1}{4}''$.
(a) Given that the radius of the spill is increasing at a rate 0.6ft/min when the radius equals 100 ft. determine the value of g.
(b)If the thickness of the oil is doubled, how does the rate of increase of the radius change?

(a) From #1,

$V'(t) = \frac{\pi}{48}2r(t)r'(t) = \frac{\pi}{24}r(t)r'(t)$,

so $\frac{g}{7.5} = \frac{\pi}{24}(100)(.6) = 2.5\pi$,

so $g = (7.5)(2.5)\pi$

$= 18.75\pi \approx 58.905$ gal/min.

(b) If the thickness is doubled, then the rate of change of the radius is halved.

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| 4- Assume the infected area of an injury is circular. (a) If the radius of the infected area is 3 mm and growing at a rate of 1 mm/hr, at what rate is the infected area increasing? (b) Find the rate of increase of the infected area when the radius reaches 6 mm. Explain in commonsense terms why this rate is larger than that of part (a). |

(a) $t =$ hours elapsed since injury
$r =$ radius of the infected area
$A =$ area of the infection
$A = \pi r^2$
$A'(t) = 2\pi r(t) \cdot r'(t)$
When $r = 3$ mm, $r' = 1$ mm/hr,
$A' = 2\pi(3)(1) = 6\pi$ mm$^2$/hr

(b) We have $A'(t) = 2\pi r r'(t)$, and $r'(t) = 1$ mm/hr, so when the radius is 6 mm we have
$A'(t) = 2\pi \cdot 6 \cdot 1 = 12\pi\,\text{mm}^2/\text{hr}$.
This rate is larger when the radius is larger because the area is changing by the same amount along the entire circumference of the circle. When the radius is larger, there is more circumerence, so the same change in radius causes a larger change in area.

| |
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| 5- Suppose that a raindrop evaporates in such a way that it maintains a spherical shape. Given that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$ and its surface area is $A = 4\pi r^2$, if the radius changes in time, show that $V' = Ar'$. If the rate of evaporation ($V'$) is proportional to the surface area, show that the radius changes at a constant rate. |

$V(t) = \frac{4}{3}\pi[r(t)]^3$
$V'(t) = 4\pi[r(t)]^2 r'(t) = Ar'(t)$
If $V'(t) = kA(t)$, then
$r'(t) = \frac{V'(t)}{A(t)} = \frac{kA(t)}{A(t)} = k.$

| |
|---|
| 6- Suppose a forest fire spreads in a circle with radius changing at a rate of 5 ft/min. When the radius reaches 200 feet, at what rate is the area of the burning region increasing? |

We have $A'(t) = 2\pi r r'(t)$, and $r'(t) = 5$ ft/min, so when the radius is 200 ft we have
$A'(t) = 2\pi \cdot 200 \cdot 5 = 2,000\pi\,\text{ft}^2/\text{min}$.

| |
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| 7- A 10 ft ladder leans against the side of a building as in example 8.2 .If the bottom of the ladder is pulled away from the wall at the rate of 3 ft/sec and the ladder remains in contact with the wall, (a) find the rate at which the top of the ladder is dropping when the bottom is 6 ft from the wall. (b) Find the rate at which the angle between the ladder and the horizontal is changing when the bottom of the ladder is 6 ft from the wall. |

(a) $10^2 = x^2 + y^2$

$$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$
$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$$
$$= -\frac{6}{8}(3)$$
$$= -2.25 \text{ ft/s}$$

(b) We have
$\cos\theta(t) = \frac{x(t)}{10}$.
Differentiating with respect to $t$ gives
$-\sin\theta(t)\cdot\theta'(t) = \frac{x'(t)}{10}$.
When the bottom is 6 feet from the wall, the top of the ladder is 8 feet from the floor and this distance is the opposite side of the triangle from *theta*. Thus, at this point, $\sin\theta = 8/10$. So
$-\frac{8}{10}\theta'(t) = \frac{3}{10}$
$\theta'(t) = -\frac{3}{8}$ rad/s.

| Q | Learning Outcome*** | Example/Exercise | Page |
|---|---|---|---|
| 13 | Solve economical and scientist problems on extrema | Example - 1 | 307 |

Example -1- Suppose that $C(x) = 0.02x^2 + 2x + 4000$ is the total cost (in AED) for a company to produce x unites of a certain product. Compute the marginal cost at x+100 and compare this to the actual of producing the 100th unit.

**Solution** The marginal cost function is the derivative of the cost function:

$$C'(x) = 0.04x + 2$$

and so, the marginal cost at $x = 100$ is $C'(100) = 4 + 2 = 6$ AED per unit. On the other hand, the actual cost of producing item number 100 would be $C(100) - C(99)$. (Why?) We have

$$\begin{aligned} C(100) - C(99) &= 200 + 200 + 4000 - (196.02 + 198 + 4000) \\ &= 4400 - 4394.02 = 5.98 \text{ AED.} \end{aligned}$$

Note that this is very close to the marginal cost of AED 6. Also notice that the marginal cost is easier to compute. ■

| Q | Learning Outcome*** | Example/Exercise | Page |
|---|---|---|---|
| 14 | Find the antiderivative of a given function | 13-15-16-23 | 329 |
| 15 | Find the antiderivative of a given function | 21-25 | 329 |

13- Find the general antiderivative $\int 2secx\ tanx\ dx$

$$\int 2\sec x \tan x dx = 2\sec x + c$$

15- Find the general antiderivative $\int 5sec^2\ x\ dx$

$$\int 5\sec^2 x dx = 5\tan x + c$$

16- Find the general antiderivative $\int 4\ \frac{cosx}{sin^2x} dx$

$$\int \frac{4\cos x}{\sin^2 x} dx = -4\csc x + c$$

23- Find the general antiderivative $\int \frac{cosx}{sinx} dx$

$$\int \frac{\cos x}{\sin x} dx = \ln|\sin x| + c$$

21- Find the general antiderivative $\int \frac{4x}{x^2+4} dx$

$$\int \frac{4x}{x^2+4} dx = 2\ln|x^2+4| + c$$

25- Find the general antiderivative $\int \frac{e^x}{e^x+3} dx$

$$\int \frac{e^x}{e^x+3} dx = \ln|e^x+3| + c$$

| Q | Learning Outcome*** | Example/Exercise | Page |
|---|---|---|---|
| 16 | Understand the notion of indefinite integral as an finding an antiderivative | 45 to 48 | 330 |

**45- Determine the position function if the velocity function is v(t)=3-12t and the initial position is s(0)=3**

Position is the antiderivative of velocity,
$s(t) = 3t - 6t^2 + c.$
Since $s(0) = 3$, we have $c = 3$. Thus,
$s(t) = 3t - 6t^2 + 3.$

**46- Determine the position function if the velocity function is $v(t) = 3e^{-t} - 2$ and the initial position is s(0)=0**

Position is the antiderivative of velocity,
$s(t) = -3e^{-t} - 2t + c.$
Since $s(0) = 0$, we have $-3 + c = 0$ and therefore $c = 3$. Thus,
$s(t) = -3e^{-t} - 2t + 3.$

**47- Determine the position function if the acceleration function is $a(t) = 3sint + 1$, the initial velocity v(0)=0 and the initial position is s(0)=4**

First we find velocity, which is the antiderivative of acceleration,
$v(t) = -3\cos t + c_1.$
Since $v(0) = 0$ we have
$-3 + c_1 = 0$, $c_1 = 3$ and
$v(t) = -3\cos t + 3.$
Position is the antiderivative of velocity,
$s(t) = -3\sin t + 3t + c_2.$
Since $s(0) = 4$, we have $c_2 = 4$. Thus,
$s(t) = -3\sin t + 3t + 4.$

**48- Determine the position function if the acceleration function is $a(t) = t^2 + 1$, the initial velocity v(0)=4 and the initial position is s(0)=0**

First we find velocity, which is the antiderivative of acceleration,
$v(t) = \frac{1}{3}t^3 + t + c_1.$
Since $v(0) = 4$ we have $c_1 = 4$ and
$v(t) = \frac{1}{3}t^3 + t + 4.$

Position is the antiderivative of velocity,
$s(t) = \frac{1}{12}t^4 + \frac{1}{2}t^2 + 4t + c_2.$
Since $s(0) = 0$, we have $c_2 = 0$. Thus,
$s(t) = \frac{1}{12}t^4 + \frac{1}{2}t^2 + 4t.$

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|---|---|---|---|
| 17 | Use the sigma notation to compute basic summation | 6-8-16 | 337 |

6- Write out all terms and compute the sums $\sum_{i=3}^{7}(i^2+i)$

$$\sum_{i=3}^{7} i^2+i = 12+20+30+42+56$$
$$= 160$$

8- Write out all terms and compute the sums $\sum_{i=6}^{8}(i^2+2)$

$$\sum_{i=6}^{8}(i^2+2)$$
$$= (6^2+2)+(7^2+2)+(8^2+2)$$
$$= 38+51+66 = 155$$

16- Use summation rules to compute the sums $\sum_{i=4}^{20}(i-3)(i+3)$

$$\sum_{i=4}^{20}(i-3)(i+3) = \sum_{i=4}^{20}(i^2-9)$$
$$= \sum_{i=4}^{20} i^2 - 9\sum_{i=4}^{20} 1$$
$$= \sum_{i=1}^{20} i^2 - \sum_{i=1}^{3} i^2 - 9\sum_{i=4}^{20} 1$$
$$= \frac{20(21)(41)}{6} - 1 - 4 - 9 - 9(17)$$
$$= 2703$$

| Q | Learning Outcome*** | Example/Exercise | Page |
|---|---|---|---|
| 18 | Estimate the area under a curve on a given interval using rectangles | 35 to 38 | 345 |

35- Use the given function values to estimate the area under the curve using left-endpoint and right-endpoint evaluation

| x | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 |
|---|---|---|---|---|---|---|---|---|---|
| F(x) | 2 | 2.4 | 2.6 | 2.7 | 2.6 | 2.4 | 2 | 1.4 | 0.6 |

Using left hand endpoints:
$L_8 = [f(0.0)+f(0.1)+f(0.2)+f(0.3)+f(0.4)+ f(0.5)+f(0.6)+f(0.7)](0.1)$
$= (2.0+2.4+2.6+2.7+2.6+2.4+2.0+ 1.4)(0.1) = 1.81$

Right endpoints:
$R_8 = [f(0.1)+f(0.2)+f(0.3)+f(0.4)+f(0.5)+ f(0.6)+f(0.7)+f(0.8)](0.2)$
$= (2.4+2.6+2.7+2.6+2.4+2.0+1.4+ 0.6)(0.1) = 1.67$

36- Use the given function values to estimate the area under the curve using left-endpoint and right-endpoint evaluation

| x | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1 | 1.2 | 1.4 | 1.6 |
|---|---|---|---|---|---|---|---|---|---|
| F(x) | 2 | 2.2 | 1.6 | 1.4 | 1.6 | 2 | 2.2 | 2.4 | 2 |

Using left hand endpoints:
$L_8 = [f().0)+f(0.2)+f(0.4)+f(0.6)+f(0.8)+ f(1.0)+f(1.2)+f(1.4)](0.2)$
$= (2.0+2.2+1.6+1.4+1.6+2.0+2.2+ 2.4)(0.2) = 3.08$

Right endpoints:
$R_8 = [f(0.2)+f(0.4)+f(0.6)+f(0.8)+f(1.0)+ f(1.2)+f(1.4)+f(1.6)](0.2)$
$= (2.2+1.6+1.4+1.6+2.0+2.2+2.4+ 2.0)(0.2) = 3.08$

37- Use the given function values to estimate the area under the curve using left-endpoint and right-endpoint evaluation

| x | 1 | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 | 1.6 | 1.7 | 1.8 |
|---|---|---|---|---|---|---|---|---|---|
| F(x) | 1.8 | 1.4 | 1.1 | 0.7 | 1.2 | 1.4 | 1.8 | 2.4 | 2.6 |

Using left hand endpoints:
$L_8 = [f(1.0)+f(1.1)+f(1.2)+f(1.3)+f(1.4)+ f(1.5)+f(1.6)+f(1.7)](0.1)$
$= (1.8+1.4+1.1+0.7+1.2+1.4+1.82+ 2.4)(0.1) = 1.182$

Right endpoints:
$R_8 = [f(1.1)+f(1.2)+f(1.3)+f(1.4)+f(1.5)+ f(1.6)+f(1.7)+f(1.8)](0.1)$
$= (1.4+1.1+0.7+1.2+1.4+1.82+2.4+ 2.6)(0.1) = 1.262$

38- Use the given function values to estimate the area under the curve using left-endpoint and right-endpoint evaluation

| x | 1 | 1.2 | 1.4 | 1.6 | 1.8 | 2 | 2.2 | 2.4 | 2.6 |
|---|---|---|---|---|---|---|---|---|---|
| F(x) | 0 | 0.4 | 0.6 | 0.8 | 1.2 | 1.4 | 1.2 | 1.4 | 1 |

Using left hand endpoints:
$L_8 = [f(1.0)+f(1.2)+f(1.4)+f(1.6)+f(1.8)+ f(2.0)+f(2.2)+f(2.4)](0.2)$
$= (0.0+0.4+0.6+0.8+1.2+1.4+1.2+ 1.4)(0.2) = 1.40$

Right endpoints:
$R_8 = [f(1.2)+f(1.4)+f(1.6)+f(1.8)+f(2.0)+ f(2.2)+f(2.4)+f(2.6)](0.2)$
$= (0.4+0.6+0.8+1.2+1.4+1.2+1.4+ 1.0)(0.2) = 1.60$

| Q | Learning Outcome*** | Example/Exercise | Page |
|---|---|---|---|
| 19 | Understand the notion of a definite integral | 15 to 18 | 356 |
| 20 | Apply the Integral Mean Value Theorem | 25 to 28 | 356 |
| 21 | Learn the properties of definite integrals | 35-36 | 356 |
| 22 | Learn the properties of definite integrals | 37-38 | 356 |

15- Write the given (total) area as an integral or sum of integrals.
The area above the x-axis and below $y = 4 - x^2$

Notice that the graph of $y = 4 - x^2$ is above the $x$-axis between $x = -2$ and $x = 2$:

$$\int_{-2}^{2} (4 - x^2)dx$$

16- Write the given (total) area as an integral or sum of integrals.
The area above the x-axis and below $y = 4x - x^2$

Notice that the graph of $y = 4x - x^2$ is above the $x$-axis between $x = 0$ and $x = 4$:

$$\int_{0}^{4} (4x - x^2)dx$$

17- Write the given (total) area as an integral or sum of integrals.
The area below the x-axis and above $y = x^2 - 4$

Notice that the graph of $y = x^2 - 4$ is below the $x$-axis between $x = -2$ and $x = 2$. Since we are asked for area and the area in question is below the $x$-axis, we have to be a bit careful.

$$\int_{-2}^{2} -(x^2 - 4)dx$$

18- Write the given (total) area as an integral or sum of integrals.
The area below the x-axis and above $y = x^2 - 4x$

Notice that the graph of $y = x^2 - 4x$ is below the $x$-axis between $x = 0$ and $x = 4$. Since we are asked for area and the area in question is below the $x$-axis, we have to be a bit careful.

$$\int_{0}^{4} -(x^2 - 4x)dx$$

25- Compute the average value of the function on the given interval.
$y = 2x + 1$ , $[0,4]$

$$f_{ave} = \frac{1}{4}\int_{0}^{4} (2x+1)dx$$
$$= \frac{1}{4}\lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n}\left(\frac{8i}{n} + 1\right)$$
$$= \lim_{n\to\infty} \left(\frac{8n(n+1)}{2n^2} + 1\right)$$
$$= 4 + 1 = 5$$

26- Compute the average value of the function on the given interval.
$y = x^2 + 2x$ , [0,1]

$$f_{ave} = \frac{1}{1}\int_0^1 (x^2+2x)dx$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left(\frac{i^2}{n^2}+\frac{2i}{n}\right)$$
$$= \lim_{n\to\infty}\left(\frac{n(n+1)(2n+1)}{6n^3}+\frac{2n(n+1)}{n^2}\right)$$
$$= \frac{2}{6}+2 = \frac{7}{3}$$

27- Compute the average value of the function on the given interval.
$y = x^2 - 1$ , [1,3]

$$f_{ave} = \frac{1}{3-1}\int_1^3 (x^2-1)dx$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left[\left(1+\frac{2i}{n}\right)^2-1\right]$$
$$= \lim_{n\to\infty}\frac{1}{n}\left(\frac{4i}{n}+\frac{4i^2}{n^2}\right)$$
$$= \lim_{n\to\infty}\left(\frac{4n(n+1)}{2n^2}+\frac{4n(n+1)(2n+1)}{6n^3}\right)$$
$$= 2+\frac{4}{3} = \frac{10}{3}$$

28- Compute the average value of the function on the given interval.
$y = 2x - 2x^2$ , [0,1]

$$f_{ave} = \frac{1}{1-0}\int_0^1 (2x-2x^2)dx$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left[2\left(\frac{i}{n}\right)-2\left(\frac{i}{n}\right)^2\right]$$
$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left(\frac{2i}{n}+\frac{2i^2}{n^2}\right)$$
$$= \lim_{n\to\infty}\left(\frac{2n(n+1)}{2n^2}+\frac{2n(n+1)(2n+1)}{6n^3}\right)$$
$$= 1+\frac{2}{3} = \frac{5}{3}$$

35- Use Theorem 4.2 to write the expression as a single integral.
(a) $\int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx$

(a) $\int_0^2 f(x)dx + \int_2^3 f(x)dx = \int_0^3 f(x)dx$

35- Use Theorem 4.2 to write the expression as a single integral.
(b) $\int_0^3 f(x)\,dx - \int_2^3 f(x)\,dx$

(b) $\int_0^3 f(x)dx - \int_2^3 f(x)dx = \int_0^2 f(x)dx$

36- Use Theorem 4.2 to write the expression as a single integral.

(a) $\int_0^2 f(x)\,dx + \int_2^1 f(x)\,dx$

(a) $\int_0^2 f(x)dx + \int_2^1 f(x)dx = \int_0^1 f(x)dx$

36- Use Theorem 4.2 to write the expression as a single integral.

(b) $\int_{-1}^2 f(x)\,dx + \int_2^3 f(x)\,dx$

(b) $\int_{-1}^2 f(x)dx + \int_2^3 f(x)dx = \int_{-1}^3 f(x)dx$

37- Assume that $\int_1^3 f(x)\,dx = 3$ and $\int_1^3 g(x)\,dx = -2$ .

(a) $\int_1^3 [f(x) + g(x)]\,dx$

(a) $\int_1^3 (f(x) + g(x))\,dx$

$= \int_1^3 f(x)\,dx + \int_1^3 g(x)\,dx$

$= 3 + (-2) = 1$

37- Assume that $\int_1^3 f(x)\,dx = 3$ and $\int_1^3 g(x)\,dx = -2$ .

(b) $\int_1^3 [2f(x) - g(x)]\,dx$

(b) $\int_1^3 (2f(x) - g(x))\,dx$

$= 2\int_1^3 f(x)\,dx - \int_1^3 g(x)\,dx$

$= 2(3) - (-2) = 8$

38- Assume that $\int_1^3 f(x)\,dx = 3$ and $\int_1^3 g(x)\,dx = -2$ .

(a) $\int_1^3 [f(x) - g(x)]\,dx$

(a) $\int_1^3 (f(x) - g(x))\,dx$

$= \int_1^3 f(x)\,dx - \int_1^3 g(x)\,dx$

$= 3 - (-2) = 5$

38- Assume that $\int_1^3 f(x)\,dx = 3$ and $\int_1^3 g(x)\,dx = -2$ .

(b) $\int_1^3 [4g(x) - 3f(x)]\,dx$

(b) $\int_1^3 (4g(x) - 3f(x))\,dx$

$= 4\int_1^3 g(x)\,dx - 3\int_1^3 f(x)\,dx$

$= 4(-2) - 3(3) = -17$

| Q | Learning Outcome*** | Example/Exercise | Page |
|---|---|---|---|
| 23 | Learn the Fundamental Theorem of Calculsu (Part I) and use it to compute various definite integrals | 1 to 6 | 366 |
| 24 | Learn the Fundamental Theorem of Calculus (Part II) and use it to compute derivatives of functions defined as definite integrals | 29 to 32 | 366 |

1- Use Part I of the Fundamental Theorem to compute each integral exactly .

$$\int_0^2 (2x-3)\,dx$$

$$\int_0^2 (2x-3)dx = (x^2-3x)\Big|_0^2 = -2$$

2- Use Part I of the Fundamental Theorem to compute each integral exactly .

$$\int_0^3 (x^2-2)\,dx$$

$$\int_0^3 (x^2-2)dx = \left(\frac{x^3}{3}-2x\right)\Big|_0^3 = 3$$

3- Use Part I of the Fundamental Theorem to compute each integral exactly .

$$\int_{-1}^1 (x^3+2x)\,dx$$

$$\int_{-1}^1 (x^3+2x)dx = \left(\frac{x^4}{4}+x^2\right)\Big|_{-1}^1 = 0$$

4- Use Part I of the Fundamental Theorem to compute each integral exactly .

$$\int_0^2 (x^3+3x-1)\,dx$$

$$\int_0^2 (x^3+3x-1)\,dx = \left(\frac{x^4}{4}-\frac{3x^2}{2}-x\right)\Big|_0^2 = -4$$

5- Use Part I of the Fundamental Theorem to compute each integral exactly .

$$\int_1^4 (x\sqrt{x}+\frac{3}{x})dx$$

$$\int_1^4 \left(x\sqrt{x}+\frac{3}{x}\right)dx = \left(\frac{2}{5}x^{5/2}+3\log x\right)\Big|_1^4 = \frac{2}{5}\cdot 32+3\log 4-\frac{2}{5}.1-3\log 1 = \frac{62}{5}+3\log 4$$

6- Use Part I of the Fundamental Theorem to compute each integral exactly .$\int_1^1 (4x-\frac{2}{x^2})dx$

$$\int_1^2 \left(4x-\frac{2}{x^2}\right)dx = \left(2x^2+\frac{2}{x}\right)\Big|_1^2 = 5$$

| 29- Find the derivative $f'(x)$. $f(x) = \int_{e^x}^{2-x} \sin t^2 \, dt$ |
|---|
| $f(x) = \int_{e^x}^{0} \sin t^2 dt + \int_0^{2-x} \sin t^2 dt$ , $f'(x) = -\sin e^{2x} \frac{d}{dx}(e^x) + \sin(2-x)^2 \frac{d}{dx}(2-x) = -e^x \sin e^{2x} - \sin(2-x)^2$ |

| 30 - Find the derivative $f'(x)$. $f(x) = \int_{2-x}^{xe^x} e^{2t} \, dt$ |
|---|
| $f(x) = \int_{2-x}^{0} e^{2t} dt + \int_0^{xe^x} e^{2t} dt$ , $f'(x) = -e^{2(2-x)} \frac{d}{dx}(2-x) + e^{2(xe^x)} \frac{d}{dx}(xe^x) = e^{4-2x} + e^{2xe^x}(xe^x + e^x)$ |

| 31 - Find the derivative $f'(x)$. $f(x) = \int_{x^2}^{x^3} \sin(2t) \, dt$ |
|---|
| $f(x) = \int_{x^2}^{0} \sin(2t) \, dt + \int_0^{x^3} \sin(2t) \, dt$<br>$f'(x) = -\sin(2x^2) \frac{d}{dx}(x^2)$<br>$+ \sin(2x^3) \frac{d}{dx}(x^3)$<br>$= -2x \sin(2x^2) + 3x^2 \sin(2x^3)$ |

| 32 - Find the derivative $f'(x)$. $f(x) = \int_{3x}^{\sin x} (t^2 + 4) \, dt$ |
|---|
| $f(x) == \int_{3x}^{0} (t^2+4) \, dt + \int_0^{\sin x} (t^2+4) \, dt$<br>$= -\int_0^{3x} (t^2+4) \, dt + \int_0^{\sin x} (t^2+4) \, dt$<br>$f'(x) = -(9x^2+4) \frac{d}{dx}(3x)$<br>$+ (\sin^2 x + 4) \frac{d}{dx}(\sin x)$<br>$= -27x^2 - 12 + \sin^2 x \cos x + 4 \cos x$ |

| Q | Learning Outcome*** | Example/Exercise | Page |
|---|---|---|---|
| 25 | Compute integrals using substitution | 11 to 19 | 376 |

11 – Evaluate the indicated integral $\int xe^{x^2+1}\,dx$

Let $u = x^2+1$ and then $du = 2x dx$ and

$$\int xe^{x^2+1}dx = \int \frac{1}{2}e^u du = \frac{1}{2}e^u + c$$
$$= \frac{1}{2}e^{x^2+1} + c$$

12 – Evaluate the indicated integral $\int e^x\sqrt{e^x+4}dx$

Let $u = e^x + 4$ and then $du = e^x dx$ and

$$\int e^x\sqrt{e^x+4}dx = \int \sqrt{u}du = \frac{2}{3}u^{3/2} + c$$
$$= \frac{1}{2}(e^x+4)^{3/2} + c$$

13 – Evaluate the indicated integral $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

Let $u = \sqrt{x}$ and then $du = \frac{1}{2\sqrt{x}}dx$ and

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = 2\int e^u du = 2e^u + c = 2e^{\sqrt{x}} + c$$

14 – Evaluate the indicated integral $\int \frac{\cos(\frac{1}{x})}{x^2}dx$

Let $u = \frac{1}{x}$ and then $du = -\frac{1}{x^2}dx$ and

$$\int \frac{\cos\left(\frac{1}{x}\right)}{x^2}dx = -\int \cos u du = -\sin u + c$$
$$= -\sin\frac{1}{x} + c$$

15 – Evaluate the indicated integral $\int \frac{\sqrt{lnx}}{x}dx$

Let $u = \ln x$ and then $du = \frac{1}{x}dx$ and

$$\int \frac{\sqrt{\ln x}}{x}dx = \int \sqrt{u}du = \frac{2}{3}u^{3/2} + c$$
$$= \frac{2}{3}(\ln x)^{3/2} + c$$

16 – Evaluate the indicated integral $\int sec^2x\sqrt{tanx}\,dx$

Let $u = \tan x$ and then $du = \sec^2 x dx$ and
Let $u = \ln x$ and then $du = \frac{1}{x}dx$ and

$$\int \sec^2 x\sqrt{\tan x}dx = \int u^{1/2}du$$
$$= \frac{2}{3}u^{3/2} + c = \frac{2}{3}(\sqrt{\tan x})^{3/2} + c$$

17 – Evaluate the indicated integral $\int \frac{1}{\sqrt{u}(\sqrt{u}+1)}du$

Let $t = \sqrt{u} + 1$ and then
$dt = \frac{1}{2}u^{-1/2}du = \frac{1}{2\sqrt{u}}du$ and

$$\int \frac{1}{\sqrt{u}\left(\sqrt{u}+1\right)}du = 2\int \frac{1}{t}dt = 2\ln|t| + c$$
$$= 2\ln\left|\sqrt{u}+1\right| + c = 2\ln\left(\sqrt{u}+1\right) + c$$

18 – Evaluate the indicated integral $\int \frac{v}{v^2+4}dv$

Let $u = v^2 + 4$ and then $du = 2vdv$ and

$$\int \frac{v}{v^2+4}dv = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + c$$
$$= \frac{1}{2}\ln\left|v^2+4\right| + c = \frac{1}{2}\ln\left(v^2+4\right) + c$$

19 – Evaluate the indicated integral $\int \frac{4}{x(lnx+1x)^2}dx$

Let $u = \ln x + 1$ and then $du = \frac{1}{x}dx$ and

$$\int \frac{4}{x(\ln x+1)^2}dx = 4\int u^{-2}du$$
$$= -4u^{-1} + c = -4(\ln x + 1)^{-1} + c$$

تراعى الحلول الأخرى

أتمنى لكم التوفيق والنجاح