Mathematicians try to convey as much information as possible with as few symbols as possible. For instance, we prefer to say $\lim_{x\to 0} \frac{1}{x^2} = \infty$ rather than $\lim_{x\to 0} \frac{1}{x^2}$ does not exist, since the first statement not only says that the limit does not exist, but also says that $\frac{1}{x^2}$ increases without bound as x approaches 0, with $x > 0$ or $x < 0$.

THEOREM 5.1

For any rational number $t > 0$,

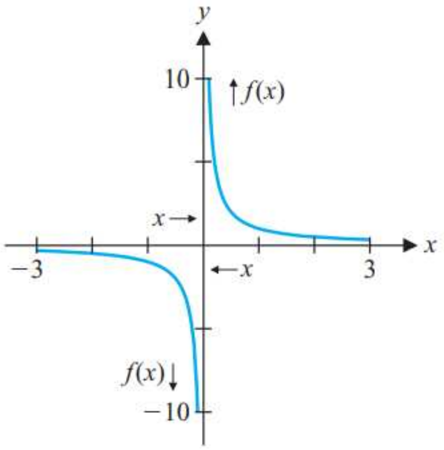
$$\lim_{x\to\pm\infty} \frac{1}{x^t} = 0,$$

where for the case where $x \to -\infty$, we assume that $t = \frac{p}{q}$, where q is odd.

THEOREM 5.2

For a polynomial of degree $n > 0$, $p_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, we have

$$\lim_{x\to\infty} p_n(x) = \begin{cases} \infty, & \text{if } a_n > 0 \\ -\infty, & \text{if } a_n < 0 \end{cases}.$$



EXAMPLE 5.1 A Simple Limit Revisited

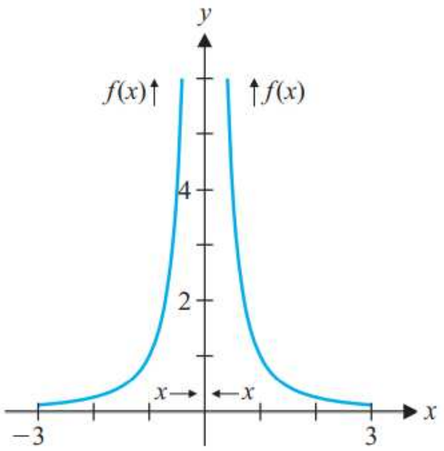
Examine $\lim_{x\to 0} \frac{1}{x}$.

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \text{ and } \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x\to 0} \frac{1}{x} \text{ does not exist.}$$

EXAMPLE 5.2 A Function Whose One-Sided Limits Are Both Infinite

Evaluate $\lim_{x\to 0} \frac{1}{x^2}$.



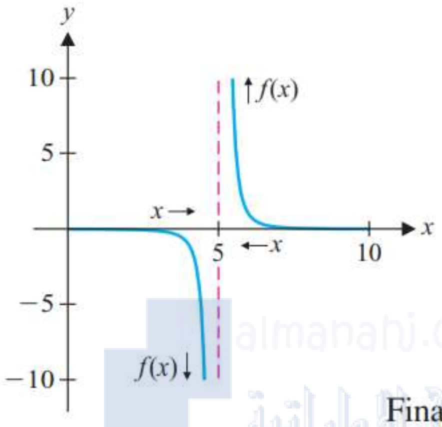
$$\lim_{x\to 0^+} \frac{1}{x^2} = \infty \quad \text{and} \quad \lim_{x\to 0^-} \frac{1}{x^2} = \infty.$$

Since both one-sided limits agree (i.e., both tend to ∞), we say that

$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

EXAMPLE 5.3 A Case Where Infinite One-Sided Limits Disagree

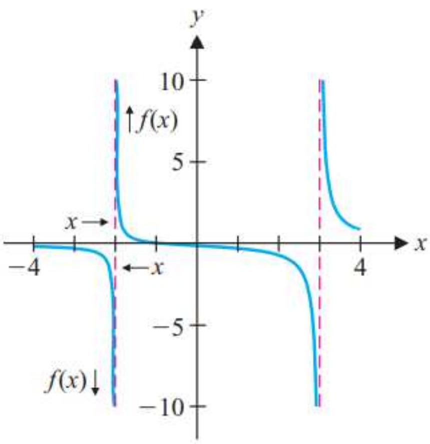
Evaluate $\lim_{x \to 5} \frac{1}{(x-5)^3}$.



$$\lim_{x \to 5^+} \frac{\overset{+}{1}}{\underset{+}{(x-5)^3}} = \infty.$$

$$\lim_{x \to 5^-} \frac{\overset{+}{1}}{\underset{-}{(x-5)^3}} = -\infty.$$

Finally, we say that $\lim_{x \to 5} \frac{1}{(x-5)^3}$ does not exist,

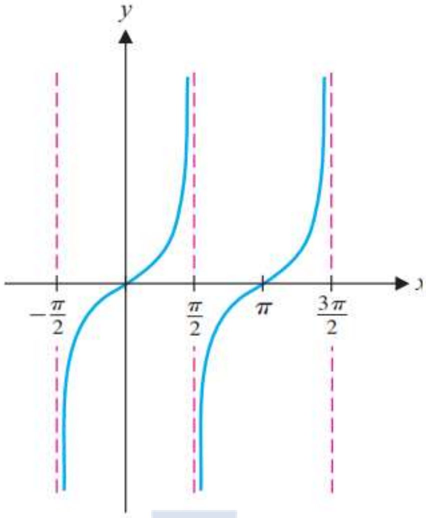


Evaluate $\lim_{x \to -2} \frac{x+1}{(x-3)(x+2)}$.

$$\lim_{x \to -2^+} \frac{\overset{-}{x+1}}{\underset{-}{(x-3)}\underset{+}{(x+2)}} = \infty$$

$$\lim_{x \to -2^-} \frac{\overset{-}{x+1}}{\underset{-}{(x-3)}\underset{-}{(x+2)}} = -\infty.$$

$\lim_{x \to -2} \frac{x+1}{(x-3)(x+2)}$ does not exist.



EXAMPLE 5.5

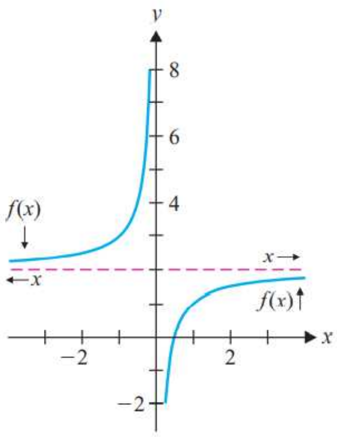
A Limit Involving a Trigonometric Function

Evaluate $\lim_{x \to \frac{\pi}{2}} \tan x$.

$$\lim_{x \to \frac{\pi}{2}^-} \tan x = \lim_{x \to \frac{\pi}{2}^-} \frac{\overset{+}{\sin x}}{\underset{+}{\cos x}} = \infty$$

$$\lim_{x \to \frac{\pi}{2}^+} \tan x = \lim_{x \to \frac{\pi}{2}^+} \frac{\overset{+}{\sin x}}{\underset{-}{\cos x}} = -\infty.$$

$\lim_{x \to \frac{\pi}{2}} \tan x$ does not exist.



EXAMPLE 5.6 Finding Horizontal Asymptotes

Find any horizontal asymptotes to the graph of $f(x) = 2 - \frac{1}{x}$.

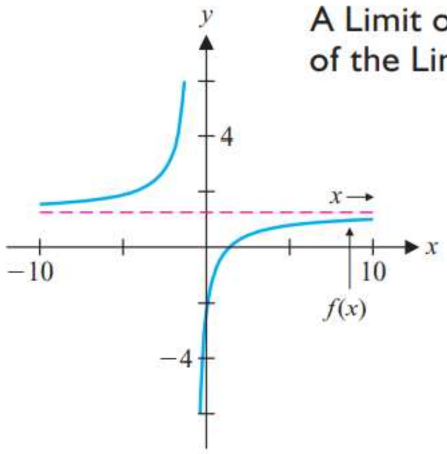
Solution We show a graph of $y = f(x)$ in Figure 1.38. Since as $x \to \pm\infty$, $\frac{1}{x} \to 0$, we get that

$$\lim_{x \to \infty} \left(2 - \frac{1}{x}\right) = 2$$

and

$$\lim_{x \to -\infty} \left(2 - \frac{1}{x}\right) = 2.$$

Thus, the line $y = 2$ is a horizontal asymptote.



A Limit of a Quotient That Is Not the Quotient of the Limits

Evaluate $\lim_{x\to\infty} \frac{5x-7}{4x+3}$.

Solution You might be tempted to write

$$\lim_{x\to\infty} \frac{5x-7}{4x+3} = \frac{\lim_{x\to\infty}(5x-7)}{\lim_{x\to\infty}(4x+3)} = \frac{\infty}{\infty} = 1.$$

EXAMPLE 5.8 Finding Slant Asymptotes

Evaluate $\lim_{x\to\infty} \frac{4x^3+5}{-6x^2-7x}$ and find any slant asymptotes.

we perform a long division:

$$\frac{4x^3+5}{-6x^2-7x} = -\frac{2}{3}x + \frac{7}{9} + \frac{5+49/9x}{-6x^2-7x}.$$

Since the third term in this expansion tends to 0 as $x \to \infty$, the function values approach those of the linear function

$$-\frac{2}{3}x + \frac{7}{9},$$

as $x \to \infty$. For this reason, we say that the graph has a **slant (or oblique) asymptote.** That is, instead of approaching a vertical or horizontal line, as happens with vertical or horizontal asymptotes, the graph is approaching the slanted straight line $y = -\frac{2}{3}x + \frac{7}{9}$.

Evaluate $\lim_{x\to 0^-} e^{1/x}$ and $\lim_{x\to 0^+} e^{1/x}$.

Solution A computer-generated graph is shown in Figure 1.41a. Although it is an unusual looking graph, it appears that the function values are approaching 0, as x approaches 0 from the left and tend to infinity as x approaches 0 from the right. To verify this, recall that $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x\to -\infty} e^x = 0$. (See Figure 1.41b for a graph of $y = e^x$.) Combining these results, we get

$$\lim_{x\to 0^-} e^{1/x} = 0.$$

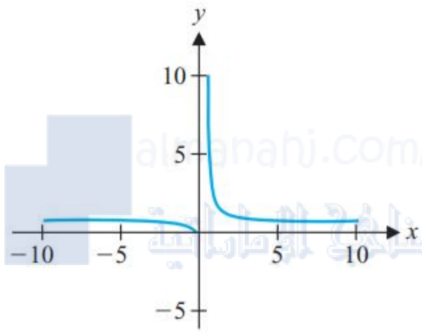


FIGURE 1.41a
$y = e^{1/x}$.

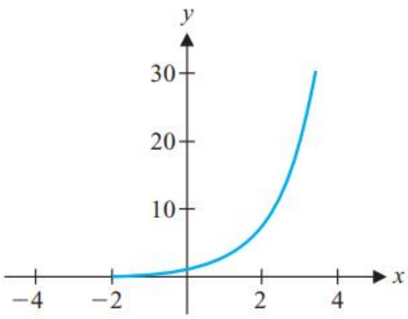


FIGURE 1.41b
$y = e^x$.

Similarly, $\lim_{x\to 0^+} \frac{1}{x} = \infty$ and $\lim_{x\to \infty} e^x = \infty$, so that

EXAMPLE 5.10 Two Limits of an Inverse Trigonometric Function

Evaluate $\lim_{x\to \infty} \tan^{-1} x$ and $\lim_{x\to -\infty} \tan^{-1} x$.

Solution The graph of $y = \tan^{-1} x$ (shown in Figure 1.42a) suggests a horizontal asymptote of about $y = -1.5$ as $x \to -\infty$ and about $y = 1.5$ as $x \to \infty$. We can be more precise with this, as follows. For $\lim_{x\to\infty} \tan^{-1} x$, we are looking for the angle that θ must approach, with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, such that $\tan\theta$ tends to ∞. Referring to the graph of $y = \tan x$ in Figure 1.42b, we see that $\tan x$ tends to ∞ as x approaches $\frac{\pi}{2}^-$. Likewise, $\tan x$ tends to $-\infty$ as x approaches $-\frac{\pi}{2}^+$, so that

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2} \quad \text{and} \quad \lim_{x\to -\infty} \tan^{-1} x = -\frac{\pi}{2}.$$

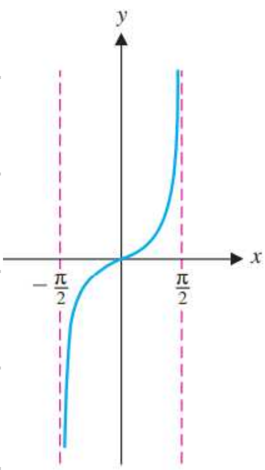


FIGURE 1.42b
$y = \tan x$

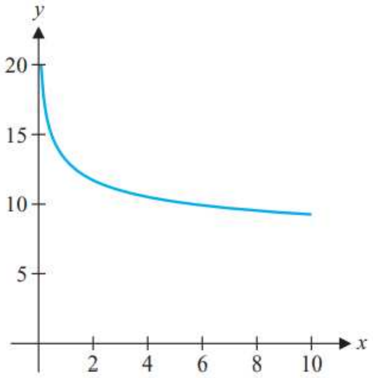


FIGURE 1.43a
$y = f(x)$

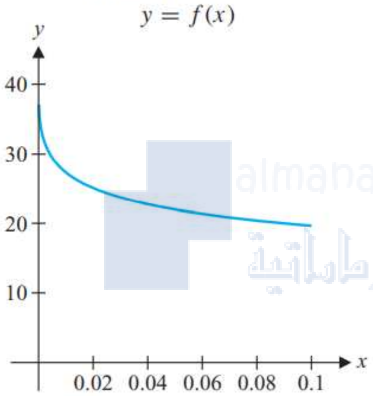


FIGURE 1.43b
$y = f(x)$

EXAMPLE 5.11 Finding the Size of an Animal's Pupils

Suppose that the diameter of an animal's pupils is given by $f(x)$ mm, where x is the intensity of light on the pupils. If $f(x) = \dfrac{160x^{-0.4} + 90}{4x^{-0.4} + 15}$, find the diameter of the pupils with (a) minimum light and (b) maximum light.

Solution Since $f(0)$ is undefined, we consider the limit of $f(x)$ as $x \to 0^+$ (since x cannot be negative). A computer-generated graph of $y = f(x)$ with $0 \le x \le 10$ is shown in Figure 1.43a. It appears that the y-values approach 20 as x approaches 0. We multiply numerator and denominator by $x^{0.4}$, to eliminate the negative exponents, so that

$$\lim_{x\to 0^+} \frac{160x^{-0.4} + 90}{4x^{-0.4} + 15} = \lim_{x\to 0^+} \frac{160x^{-0.4} + 90}{4x^{-0.4} + 15} \cdot \frac{x^{0.4}}{x^{0.4}}$$

$$= \lim_{x\to 0^+} \frac{160 + 90x^{0.4}}{4 + 15x^{0.4}} = \frac{160}{4} = 40 \text{ mm}.$$

This limit does not seem to match our graph. However, in Figure 1.43b, we have zoomed in so that $0 \le x \le 0.1$, making a limit of 40 looks more reasonable.

For part (b), we consider the limit as x tends to ∞. From Figure 1.43a, it appears that the graph has a horizontal asymptote at a value somewhat below $y = 10$. We compute the limit

$$\lim_{x\to\infty} \frac{160x^{-0.4} + 90}{4x^{-0.4} + 15} = \frac{90}{15} = 6 \text{ mm}.$$

So, the pupils have a limiting size of 6 mm, as the intensity of light tends to ∞.

In exercises 5–22, determine each limit (answer as appropriate, with a number, ∞, $-\infty$ or does not exist).

5. $\lim_{x \to -2} \frac{x^2 + 2x - 1}{x^2 - 4}$

6. $\lim_{x \to -1^-} (x^2 - 2x - 3)^{-2/3}$

7. $\lim_{x \to 0} \cot x$

8. $\lim_{x \to \pi/2} x \sec^2 x$

9. $\lim_{x \to \infty} \frac{x^2 + 3x - 2}{3x^2 + 4x - 1}$

10. $\lim_{x \to \infty} \frac{2x^2 - x + 1}{4x^2 - 3x - 1}$

11. $\lim_{x \to -\infty} \frac{-x}{\sqrt{4+x^2}}$

12. $\lim_{x \to \infty} \frac{2x^2-1}{4x^3-5x-1}$

13. $\lim_{x \to \infty} \ln\left(\frac{x^2+1}{x-3}\right)$

14. $\lim_{x\to 0^+} \ln(x \sin x)$

15. $\lim_{x\to 0^+} e^{-2/x^3}$

16. $\lim_{x\to\infty} e^{-(x+1)/(x^2+2)}$

17. $\lim_{x \to \infty} \cot^{-1} x$

18. $\lim_{x \to \infty} \sec^{-1} \dfrac{x^2 + 1}{x + 1}$

19. $\lim_{x \to 0} \sin(e^{-1/x^2})$

20. $\lim_{x \to \infty} \sin(\tan^{-1} x)$

21. $\lim_{x \to \pi/2} e^{-\tan x}$

22. $\lim_{x \to 0^+} \tan^{-1}(\ln x)$

In exercises 23–28, determine all horizontal and vertical asymptotes. For each side of each vertical asymptote, determine whether $f(x) \to \infty$ or $f(x) \to -\infty$.

23. (a) $f(x) = \dfrac{x}{4 - x^2}$ (b) $f(x) = \dfrac{x^2}{4 - x^2}$

24. (a) $f(x) = \dfrac{x}{\sqrt{4 + x^2}}$ (b) $f(x) = \dfrac{x}{\sqrt{4 - x^2}}$

25. $f(x) = \dfrac{3x^2+1}{x^2-2x-3}$

26. $f(x) = \dfrac{1-x}{x^2+x-2}$

27. $f(x) = 4\tan^{-1}x - 1$

28. $f(x) = \ln(1 - \cos x)$

In exercises 29–32, determine all vertical and slant asymptotes.

29. $y = \dfrac{x^3}{4 - x^2}$

30. $y = \dfrac{x^2 + 1}{x - 2}$

31. $y = \dfrac{x^3}{x^2 + x - 4}$

32. $y = \dfrac{x^4}{x^3 + 2}$

33. Suppose that the size of the pupil of a certain animal is given by $f(x)$ (mm), where x is the intensity of the light on the pupil. If $f(x) = \dfrac{80x^{-0.3} + 60}{2x^{-0.3} + 5}$, find the size of the pupil with no light and the size of the pupil with an infinite amount of light.

36. Find a function of the form $f(x) = \dfrac{20x^{-0.4} + 16}{g(x)}$ such that $\lim\limits_{x \to 0^+} f(x) = 5$ and $\lim\limits_{x \to \infty} f(x) = 4$.

51. Suppose that $f(x)$ is a rational function $f(x) = \dfrac{p(x)}{q(x)}$ with the degree of $p(x)$ greater than the degree of $q(x)$. Determine whether $y = f(x)$ has a horizontal asymptote.

52. Suppose that $f(x)$ is a rational function $f(x) = \dfrac{p(x)}{q(x)}$ with the degree (largest exponent) of $p(x)$ less than the degree of $q(x)$. Determine the horizontal asymptote of $y = f(x)$.

53. Suppose that $f(x)$ is a rational function $f(x) = \dfrac{p(x)}{q(x)}$. If $y = f(x)$ has a slant asymptote $y = x + 2$, how does the degree of $p(x)$ compare to the degree of $q(x)$?

54. Suppose that $f(x)$ is a rational function $f(x) = \dfrac{p(x)}{q(x)}$. If $y = f(x)$ has a horizontal asymptote $y = 2$, how does the degree of $p(x)$ compare to the degree of $q(x)$?

55. Find a quadratic function $q(x)$ such that $f(x) = \dfrac{x^2 - 4}{q(x)}$ has one horizontal asymptote $y = -\frac{1}{2}$ and exactly one vertical asymptote $x = 3$.

56. Find a quadratic function $q(x)$ such that $f(x) = \dfrac{x^2 - 4}{q(x)}$ has one horizontal asymptote $y = 2$ and two vertical asymptotes $x = \pm 3$.

57. Find a function $g(x)$ such that $f(x) = \dfrac{x^3 - 3}{g(x)}$ has no vertical asymptotes and has a slant asymptote $y = x$.

58. Find a function $g(x)$ such that $f(x) = \dfrac{x - 4}{g(x)}$ has two horizontal asymptotes $y = \pm 1$ and no vertical asymptotes.

In exercises 59–64, label the statement as true or false (not always true) for real numbers *a* and *b*.

59. If $\lim_{x\to\infty} f(x) = a$ and $\lim_{x\to\infty} g(x) = b$, then $\lim_{x\to\infty} [f(x) + g(x)] = a + b$.

60. If $\lim_{x\to\infty} f(x) = a$ and $\lim_{x\to\infty} g(x) = b$, then $\lim_{x\to\infty} \left[\frac{f(x)}{g(x)}\right] = \frac{a}{b}$.

61. If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty} [f(x) - g(x)] = 0$.

62. If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty} [f(x) + g(x)] = \infty$.

63. If $\lim_{x\to\infty} f(x) = a$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty} \left[\frac{f(x)}{g(x)}\right] = 0$.

64. If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty} \left[\frac{f(x)}{g(x)}\right] = 1$.